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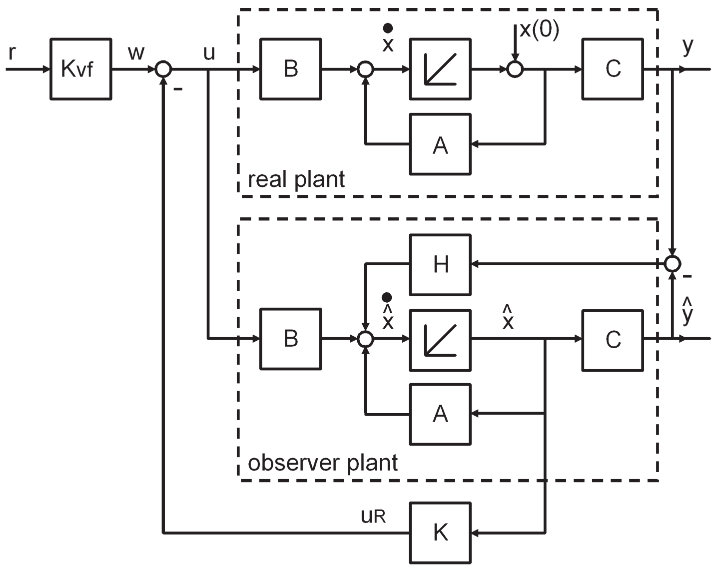
State Space Control, LQR and Observer
step by step introduction, with Matlab examples



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Introduction

State of the art regulators are of the type PID combined with filters. However, they can be found in various combinations, such as cascaded structures or loops with feed forward. The use of PID regulators leads to many advantages: On one hand, the linear theory allows to design excellent regulators for single in single out systems. On the other hand, there is huge number of existing plants, which are robust and stable for themselves and therefore, they don't need to be identificated or physically modeled. For those plants, simple rules from Ziegler and Nichols solve the regulator problems quite well, and the engineering process is very fast.

State Space regulators are not as well known by many of the engineers. Either they are a little bit more complicated to understand, either a modeling of the sytem is needed to design a robust and stable feedback loop. However, there is a variety of plants, where a use of State Space regulators can be justified. Especially, if regulators of multiple in and multiple out systems need to be designed, a little bit more complicated regulator structure causes very often very robust stable systems, which never could be reached by PID regulators.

If in addition an observer is used, it is possible to avoid the integration of additional sensors, since the model, which is working in the control loop software, can reconstruct the states by software.

This booklet introduces that theory to the engineers. The prerequisites are a knowledge about Laplace Operator, transfer function and stability criteria. Those skills are provided by any Bachelor Degree Schools of Engineering all around the world.

For deeper understanding, each chapter contains solved excercises at the end. The solutions are very often supplied by a small Matlab code. This way, it is possible to see the effects of changing parameters.

Thus it helps the engineer solving a regulator problem, to get an other view of control loop feedback and simplifies the decision, whether to choose a conventional PID structure or a State Space regulator.

Contents

Chapter 1

State Space description

1 Input Output description and State Space description

A SISO (single in - single out) system in input-output-description (also called 'transferfunction'), where U(s) is the input and Y(s) is the output is given by

$$Y(s) = \frac{b_m \cdot s^m + b_{m-1} \cdot s^{m-1} + ... + b_0}{s^n + a_{n-1} \cdot s^{n-1} + ... + a_0} \cdot U(s)$$

A system in State Space description is given by

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t) \quad \text{and} \quad y(t) = C \cdot x(t) + D \cdot u(t)$$

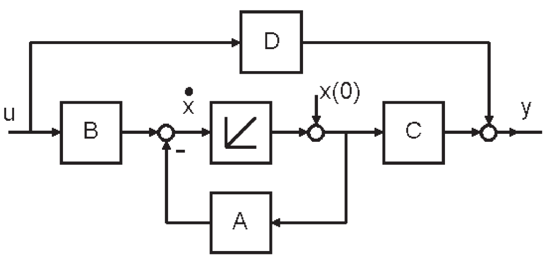


Figure 1.1: State Space description as block diagram

A	n x n Matrix	n: (number of) states
B	n x m Matrix	m: (number of) inputs
C	r x n Matrix	r: (number of) outputs
D	r x m Matrix	

A: system matrix, defines the system dynamics

B: input matrix, defines what effect the input values have on the n states of the system

C: output matrix, defines the linear combination of the states to calculate the output value(s)

D: straight-way matrix, defines, how the input values u work directly on the output values y. In the input-output-description, m = n, if D is not zero. In this case, the step response of the output contains a step component too.

2 Find the Input Output description (at a given State Space description)

Laplace transforming of a state vector x(t), assuming $x(t_0) = 0$ follows

$$sX(s) = AX(s) + BU(s)$$

and

$$X(s) = (sI - A)^{-1}BU(s)$$

the output equation is then

$$Y(s) = (C(sI - A)^{-1}B + D)U(s)$$

Thus, the transferfunction (input-output-description) is calculated to:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

3 Find the State Space description (at a given Input Output description)

The A, B, C and D matrixes are not unique for a given system. It depends the manner, how the state variables x are choosed. Using a so called similarity transformation, it is possible to take those matrixes into special forms.

$$\tilde{x}(t) = Tx(t)$$

T is either a regular n x n matrix or a scalar $\neq 0$. The components x must no longer correspond to physical values. Doing the transformation for the whole equation, we get the following

$$T^{-1}\dot{\tilde{x}}(t) = AT^{-1}\tilde{x}(t) + Bu(t) \quad \text{and} \quad y(t) = CT^{-1}\dot{\tilde{x}}(t) + Du(t)$$

This leads to new matrixes

$$\tilde{A} = TAT^{-1} \quad , \quad \tilde{B} = TB \quad , \quad \tilde{C} = CT^{-1} \quad , \quad \tilde{D} = D$$

Now, two special forms of A,B,C and D matrixes will be treated, the controllable canonical form and the observable canonical fom. See also the solved excercises at the end of the chapter.

3.1 Controllable canonical form

The general forms of A_c, B_c and C_c matrixes are as follows:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ * & * & * & * \end{pmatrix} \quad , \quad B_c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad , \quad C_c = \begin{pmatrix} * & * & * & * \end{pmatrix}$$

Assuming a system in input-output description

$$Y(s) = \frac{b_m \cdot s^m + b_{m-1} \cdot s^{m-1} + ... + b_0}{s^n + a_{n-1} \cdot s^{n-1} + ... + a_0} \cdot U(s)$$

the complete matrixes are

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix} \quad , \quad B_c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$C_c = \begin{pmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \dots & b_{n-1} - b_n a_{n-1} \end{pmatrix} \quad , \quad D_c = b_n$$

3.2 Observable canonical form

The following A_o, B_o, C_o and D_o matrixes are called observable canonical form. Assumed is the same input-output-description as above.

$$A_o = \begin{pmatrix} 0 & 0 & \dots & -a_0 \\ 1 & 0 & \dots & -a_1 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \quad , \quad B_o = \begin{pmatrix} b_0 - b_n a_0 \\ b_1 - b_n a_1 \\ \vdots \\ b_{n-1} - b_n a_{n-1} \end{pmatrix}$$

$$C_o = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad D_o = b_n$$

4 Controllability and Observability

Controllability of a system means, that there exists a finite control input u that can transfer any initial state x(0) to any other desired state x(t). It is determined whether the system is controllable or not by investigating the algebraic condition below. A System is controllable, if the determinant of D_c is not zero. Note that here the Matrix C is not involved.

$$Rank[D_c] = Rank[B, AB, A^2B...A^{n-1}B] = n$$

The ability to estimate (observe) every state variable is called observability. A system is observable if there is existing a finite time T within that the initial state x(0) can be determined from the history y(t) and additionally the given control information u(t). The system is observable if the determinant of D_o is not zero. Note that here the Matrix B is not involved.

$$Rank[D_o] = Rank[\begin{pmatrix} C \\ CA \\ \vdots \\ \\ CA^{n-1} \end{pmatrix}] = n$$

All real and imaginary parts of the roots of a system can be placed where ever desired in the s-plane by a State Space regulator if a system is observable and controllable (and only if!).
If a system is not controllable, more or different actuators are needed, which leads to a different Matrix B. If a system is not observable, more or different sensors are needed, which leads to a different Matrix C.

5 Stability

Using input-output-description a system is stable, if all poles of the transfer function have negative real parts. In this case, the impulse response g(t), is zero, for t=∞, where

$$g(t) = L^{-1}[G(s)]$$

Using State Space description a system is stable, if all the roots of the characteristic equation det(sI-A)=0 have negative real parts.

6 Solved exercises

electrical system

A electrical system in State Space description is given

$$A = \begin{pmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{pmatrix} \quad , \quad B = \begin{pmatrix} \frac{1}{C} \\ 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & R \end{pmatrix} \quad , \quad D = 0$$

Find the input-output-description G(s) = $\frac{Y(s)}{U(s)}$

model descriptions

Given is a system in input-output-description

$$Y(s) = \frac{s + 12}{s^3 + 4 \cdot s^2 + 4 \cdot s + 2} \cdot U(s)$$

1. find the State Space description (controllable canonical form)
2. is the system controllable and observable?

spring and mass

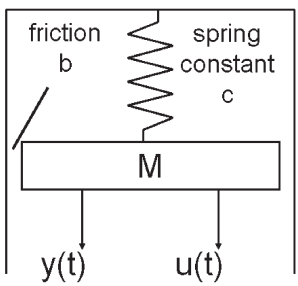


Figure 1.2: spring and mass with friction

Given is a system with spring and mass, where u(t), the input, is a force and y(t), the output, is the position of the mass.

1. find the differential equations of the system
2. find the State Space description assuming $(x_1, x_2) = (y, \dot{y})$

inverted pendulum

Given is an inverted pendulum on top of a vehicle. u(t), the input, is a force and y(t), the output, is the position of the vehicle. By

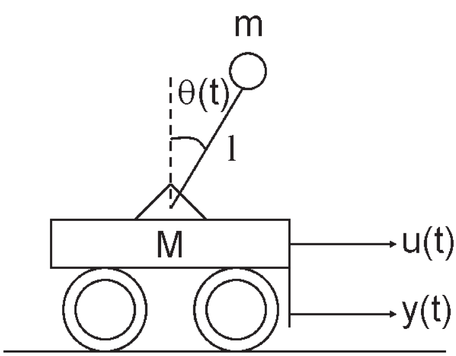


Figure 1.3: inverted pendulum

establishing a physical model of the system, the following equations can be found by linearization

$$M\ddot{y} + ml\ddot{\theta} - u = 0 \quad \text{and} \quad ml\ddot{y} + ml^2\ddot{\theta} - mlg\theta = 0$$

the state vector (x_1, x_2, x_3, x_4) is assumed as $(y, \dot{y}, \theta, \dot{\theta})$

1. find the system matrixes A,B,C and D
2. for the following considerations assume M = 1kg, m = 0.1kg, $g = 10m/s^2$, l=1m. Is the system controllable and observable ?
3. is the system stable ?

Solutions

solution electrical system

$$(sI-A)=\begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s+\frac{R}{L}\end{bmatrix}$$

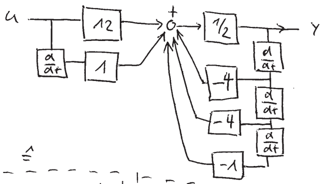
$$\Phi(s)=(sI-A)^{-1} = \frac{1}{s^2+s\cdot\frac{R}{L}+\frac{1}{LC}}\begin{bmatrix} s+\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s\end{bmatrix}$$

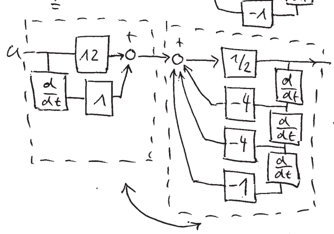
$$G(s)=C\cdot\Phi(s)\cdot B=\underbrace{\underbrace{\begin{bmatrix}0 & R\end{bmatrix}\circ\frac{1}{s^2+s\cdot\frac{R}{L}+\frac{1}{LC}}\begin{bmatrix} s+\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s\end{bmatrix}}_{\frac{1}{s^2+s\cdot\frac{R}{L}+\frac{1}{LC}}\begin{bmatrix}\frac{R}{L} & sR\end{bmatrix}}\circ\begin{bmatrix}\frac{1}{C} \\ 0\end{bmatrix}}$$

$$=\frac{1}{s^2+s\cdot\frac{R}{L}+\frac{1}{LC}}\cdot\frac{R}{LC}=\frac{\frac{R}{LC}}{s^2+s\cdot\frac{R}{L}+\frac{1}{LC}}$$

solution model descriptions

$$\frac{Y(s)}{U(s)} = \frac{s+12}{s^3+4s^2+4s+2} \qquad Y = \frac{1}{2}\left(\dot{u} + 12u - \dddot{y} - 4\ddot{y} - 4\dot{y}\right)$$





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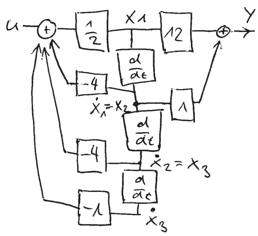


$$X_1 = \frac{1}{2}\left(u - 4x_2 - 4x_3 - 1\dot{x}_3\right)$$

$$\Rightarrow 2 \cdot x_1 = u - 4x_2 - 4x_3 - 1\dot{x}_3$$

$$\Rightarrow \dot{x}_3 = u - 4x_2 - 4x_3 - 2x_1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -4 \end{bmatrix} ; \quad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -4 & -4 \\ 8 & 16 & 12 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -4 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \circ u$$

$$Y = \begin{bmatrix} 12 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u$$

Controllability:

$$\mathrm{Det}\,(B \;\; AB \;\; A^2B) = \left\| \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 12 \end{matrix} \right\| = -1 \quad \Rightarrow \text{controllable}$$

Observability:

$$\mathrm{Det}\,(C \,;\, CA \,;\, CA^2) = \left\| \begin{matrix} 12 & 1 & 0 \\ 0 & 12 & 1 \\ -2 & -4 & -8 \end{matrix} \right\| = -1298 \quad \Rightarrow \text{observable}$$

solution spring and mass

$$M\ddot{y} + b\dot{y} + cy = u$$

$$M\dot{x}_2 + bx_2 + cx_1 = u \quad ; \quad \dot{x}_1 = x_2$$

$$\Rightarrow \dot{x}_2 = -\frac{b}{M}\cdot x_2 - \frac{c}{M}\cdot x_1 + \frac{1}{M}\cdot u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{M} & -\frac{b}{M} \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \circ u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

solution inverted pendulum

(1) $M\dot{x}_2 + ml\cdot\dot{x}_4 - u = 0$

$\Rightarrow$ (1') $\dot{x}_2 = -\frac{m}{M}\cdot l\dot{x}_4 + \frac{u}{M}$

(2) $\dot{x}_2 + l\cdot\dot{x}_4 - g\cdot x_3 = 0$

$\Rightarrow$ (2') $ml\dot{x}_4 = mgx_3 - m\dot{x}_2$

(2') in (1): $M\dot{x}_2 + mgx_3 - m\dot{x}_2 - u = 0$

$\Rightarrow (M-m)\dot{x}_2 = -mgx_3 + u$

(1') in 2: $-\frac{m}{M}\cdot l\dot{x}_4 + \frac{u}{M} + l\dot{x}_4 - gx_3 = 0$

$\Rightarrow l\left(1-\frac{m}{M}\right)\cdot\dot{x}_4 = gx_3 - \frac{u}{M}$

[num, den] = ss2tf(A,B,C,D)
roots(den)
→ 0 / 0 / -3,33 / 3,33
=> unstable!

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\frac{m}{M-m}g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{gM}{l(M-m)} & 0\end{bmatrix}}_{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -1{,}11 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 11{,}1 & 0\end{bmatrix}} \circ \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{1}{M-m}\\ 0\\ -\frac{1}{l(M-m)}\end{bmatrix}}_{\begin{bmatrix}0\\ 1{,}11\\ 0\\ -1{,}11\end{bmatrix}} \circ u; \quad y = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} \circ \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

rank(ctrb(A,B)) = 4
=> controllable

rank(obsv(A,C)) = 4
=> observable

Chapter 2

State Space regulator

1 A plant in State Space description completed with a feedback loop

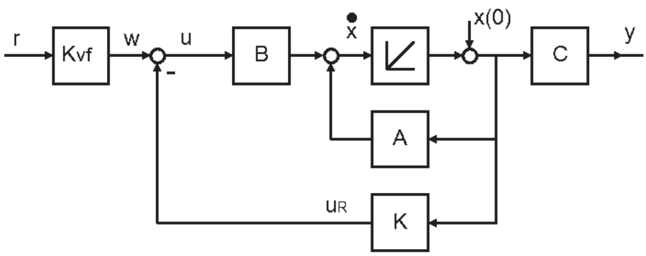


Figure 2.1: Block diagram of a State Space regulator with full state feedback

A prefilter K_{vf} is a transfer function or in most cases just a scalar. It filters the input signal R(s) in a way, that in steady state the output y has the same value as a constant input r.

The feedback of the state vector $x(t)$ is weighted by the coefficients $k_1, k_2, ..., k_n$. This is exactly the core of the State Space regulator

Furthermore, in comparision with conventional feedback-structures, the error signal $e = r - y$ is no more visible in block diagrams.

$$u_R = k_1x_1+k_2x_2+...+k_nx_n \quad \text{or} \quad u_R = \begin{pmatrix} k_1 & k_2 & ... & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = Kx$$

The control signal u is calculated by

$$u = w - u_R = K_{vf} \cdot r - K \cdot x$$

and the matrix equations

$$\dot{x} = A \cdot x + B \cdot (K_{vf} \cdot r - Kx) = A \cdot x - BKx + BK_{vf} \cdot r$$

$$\dot{x} = (A - BK) \cdot x + K_{vf}B \cdot r \quad \text{and} \quad y = C \cdot x \quad (\text{assuming D} = 0)$$

The new system (is still linear of course) can be described with new System Matrixes, A_g, B_b, C_g

$$A_g = A - BK \quad , \quad B_g = K_{vf}B \quad , \quad C_g = C$$

2 Calculation of K_{vf} and $k_1, k_2, ..., k_n$ using pole placement

See solved excercise

If the poles of the new closed-loop system are placed far (mostly left) away from the old poles of the plant, the values of $k_1, k_2, ..., k_n$ are usually large. This means, that the control output is large as well and could be limited by hardware. Note that this limitation of the output causes differences between the physical model and the real system, if it is not considered.

3 Optimal control

The key issue of a LQ-Regulator is to take care to both a limitation of the control signal u and moreover a fast transcient behaviour of the states. The integral below represents the performance index J. The optimal control design minimizes J, which contains the input u as well as the states x. It is the integral of the squared error. This must be worked out this way, because negative and positive errors need to be treated equally. Such systems are also called optimal control systems. The index J can be written as

$$J = \int_{0}^{t_e} g(x, u, t)dt$$

X is representing the state vector, u is the control vector, and t_e the end of time intervall. We are generally interested in minimizing the error. If the desired state vector is represented as $x_s = 0$, it is possible to treat the error as equal to the state vector itself. This means that any difference between the state vector and zero is equal to an error. The picture below (Fig. 2.2) shows this.

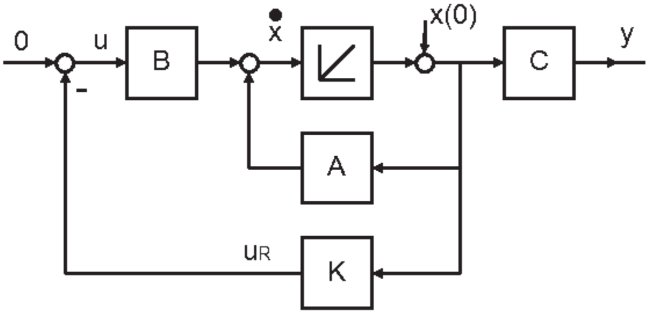


Figure 2.2: Block diagram of a LQ-Regulator

The control system can be written as

$$\dot{x} = (A - BK) \cdot x = Lx$$

L is the n x n matrix resulting from a subtraction of the elements of BK from the elements of A. Considering the fact, that negative and positive errors need to have the same effect, the following matrix operation is used. As wished, it squares and adds the states, means the errors.

$$x^T x = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = {x_1}^2 + {x_2}^2 + ... + {x_n}^2$$

Furthermore, it is possible to define a weight matrix Q to weight the states individualy. Then, the specific form of the performance index, is

$$J = \int\limits_0^{t_e} (x^T Q x) dt$$

Later on, the general form will be worked out as well. It contains also a term u, to take care to the control energy part of the performance index J. But up to now, just the influence of the state vectors, their deviation from zero will be investigated. The final time of interest is t_e $= \infty$. To minimize the value of J, we assume, that an exact differential is existing so that

$$\frac{d}{dt}(x^T S x) = -x^T Q x$$

where S has to be determined. Using the product rule for differentiation (and assuming that the S matrix is symmetric)

$$\frac{d}{dt}(x^T S x) = \dot{x}^T S x + x^T S \dot{x}$$

and since $\dot{x} = Lx$

$$\frac{d}{dt}(x^T S x) = (Lx)^T S x + x^T S(Lx) = x^T L^T S x + x^T S L x = x^T (L^T S + S L) x$$

here, $(Lx)^T = x^T L^T$. Furthermore, we let $(L^T S + SL) =$ -Q, doing so the equation gets

$$\frac{d}{dt}(x^T S x) = -x^T Q x$$

The performance index J can be found now by doing a substitution, because this is exactly the differential we are looking for.

$$J = \int\limits_0^{\infty} -\frac{d}{dt}(x^T S x) dt = 0 - (-x^T(0) S x(0))$$

or

$$J = \int_0^\infty (x^T Q x) dt = x^T(0) S x(0) \quad \text{and} \quad (L^T S + SL) = -Q$$

To minimize the performance index J, the following has to be carried out: first determining the matrix S, by a known L and Q, and second minimize J by changing the parameters of S as far as it is possible.

3.1 LQ-Regulator theory

Generally, we take also care to the control output u. Consider the following SISO system

$$\dot{x} = (A) \cdot x + Bu$$

with a feedback

$$u = -Kx = -\begin{pmatrix} k_1 & k_2 & ... & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the performance index as shown above is

$$J = \int_0^\infty (x^T Q x + r u^2) dt$$

Here, r is a scalar weighting factor, since u is also a scalar. It can be shown that this index is minimized when

$$K = r^{-1} B^T S$$

the n x n matrix S is calculated as the solution of the matrix Riccati equation

$$A^T S + SA - SBB^T S r^{-1} = -Q$$

where Q is I for equal weighting of the states (alternatively $Q = C'C$, to involve also the Zeros of the system). This optimal control, called the linear quadratic regulator can be found using the Matlab function lqr. There, r and Q can be given as parameters.

3.2 LQ-Regulator stability

It can be shown, that a LQ-regulator with full state feedback has got at least a phase margin of 60° or a gain margin of 0.5 to ∞ which means, that a LQ-regulator is a very robust controller. See also the solved exercises at the end of the chapter.

4 State Space regulator including integral part

Assuming that our plant has no integral part, also the State Space regulator, which just multiplies the states with a factor, is not able to realize any integral part. From the linear theory it is known, that this fact leads to a zero steady state error. To avoid this error, as using a normal PI regulator, an integral part needs to be worked out. If the input r is a step, the derivation of r is zero

$$\dot{r} = 0$$

the error e is

$$e = y - r$$

This relation is also valid for the time derivative

$$\dot{e} = \dot{y} = C\dot{x}$$

since $\dot{r} = 0$. Here, the following block diagram is used. If we define

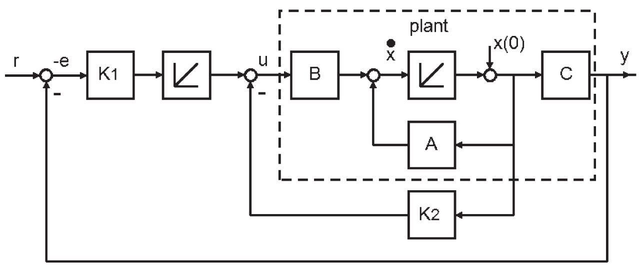


Figure 2.3: Block diagram of a regulator with integral part

two new variables, m and n, as below

$$m = \dot{x} \;\text{ and }\; n = \dot{u}$$

a new system with new matrixes can be defined.

$$\begin{pmatrix} \dot{e} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix} \begin{pmatrix} e \\ m \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} n$$

The order is higher since the integration is an additional state. Doing so the following form can be found (K1 respectively K2 can be found eg. by pole placement or optimal control)

$$n(t) = -K_1 e(t) - K_2 m(t)$$

Where K_1 is a scalar representing the integral part. K_2 ist representing the control vector. Since u ist the integral of n, the control input is

$$u(t) = -K_1 \int_0^t e(\tau) d\tau - K_2 x(t)$$

where the error e needs to be zero to avoid an infinite control input u. See also the solved exercises at the end of the chapter.

5 Solved exercises

pole placement

A system plant in State Space description is given

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 6 & 0 \end{pmatrix} \quad , \quad D = 0$$

Find K and K_{vf} for the following given poles p_1 and p_2 of the closed loop system:

1. $p_1 = p_2 =$ -12
2. $p_1 =$ -12+i·12, $p_2 =$ -12-i·12
3. $p_1 = p_2 =$ -40

optimal control

A scalar system is given

$$\dot{x} = ax + bu$$

1. find a regulator $u(x) = -kx$ by minimizing the performance index J.

   $$J = \int_0^\infty (qx^2 + ru^2)dt \quad , \quad (q, r > 0)$$

   Algebraic Riccati equation: $2ap - \frac{1}{r}b^2p^2 + q = 0$

2. find a regulator $u(x) = -kx$, which stabilizes the system with a minimum of control energy (q=0).

LQ-regulator of an inverted pendulum

Given is an inverted pendulum. u(t), the input, is a force and y(t), the output, is the position of the vehicle. The state vector (x_1, x_2, x_3, x_4)

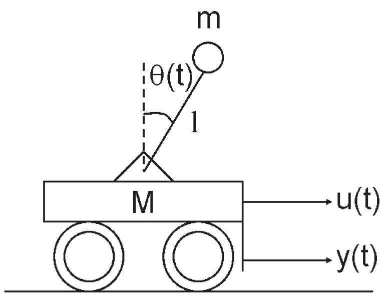


Figure 2.4: inverted pendulum

is assumed as $(y, \dot{y}, \theta, \dot{\theta})$ and M = 1kg, m = 0.1kg, $g = 10m/s^2$, l=1m. The following system matrixes can be found:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1.11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11.1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ 1.11 \\ 0 \\ -1.11 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad D = 0$$

1. compute some optimal control regulators using Matlab command lqr and plot step responses of the closed-loop systems at different weights of control energy r. Find also suitable prefilters K_{vf}.
2. for the same weight's of control energy r, plot the bode diagram of the closed-loop systems and discuss the results
3. discuss the robustness of the system

state regulator including integral part

Given is a system in input-output-description

$$Y(s) = \frac{1}{s^2 + 3 \cdot s + 3} \cdot U(s)$$

For a step input the steady-state error is not zero.

1. Find the State Space description in controllable canonical form.
2. Assme that all states are directly measurable (what would You do if not?). Add an integral part and design a controller with closed loop poles p_1=-1+i·1, p_2=-1-i·1, p_3=-12 using Matlab.
3. Plot the step response of the output y using Simulink.

Solutions

solution pole placement

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - A_g)^{-1} \cdot B_g \; ; \; A_g = A - BK \; ; \; B_g = K_{vf} \cdot B$$

$$A_g = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \end{pmatrix} (K_1 \; K_2) = \begin{pmatrix} 0 & 1 \\ -6K_1 & -3-6K_2 \end{pmatrix} ; \; B_g = \begin{pmatrix} 0 \\ K_{vf} \cdot 6 \end{pmatrix}$$

$$(sI - A_g) = \begin{pmatrix} s & -1 \\ 6K_1 & s+3+6K_2 \end{pmatrix}$$

$$(sI - A_g)^{-1} = \frac{1}{s^2 + s(3+6K_2) + 6K_1} \cdot \begin{pmatrix} s+3+6K_2 & 1 \\ -6K_1 & s \end{pmatrix}$$

$$G(s) = \frac{1}{s^2 + s(3+6K_2) + 6K_1} \circ (6 \quad 0) \circ \begin{pmatrix} s+3+6K_2 & 1 \\ -6K_1 & s \end{pmatrix} \circ \begin{pmatrix} 0 \\ K_{vf} \cdot 6 \end{pmatrix}$$

$$= \frac{1}{s^2 + s(3+6K_2) + 6K_1} \circ (6 \quad 0) \circ \begin{pmatrix} K_{vf} \cdot 6 \\ s \cdot K_{vf} \cdot 6 \end{pmatrix} = \frac{6 \cdot K_{vf} \cdot 6}{s^2 + s(3+6K_2) + 6K_1}$$

Denominator: $P(s) = (s - p_1)(s - p_2)$

$$= s^2 - (p_1 + p_2) \cdot s + p_1 \cdot p_2 \stackrel{!}{=} s^2 + s(3+6K_2) + 6K_1$$

$$\Rightarrow -p_1 - p_2 = 3 + 6K_2 \Rightarrow K_2 = -\frac{p_1 + p_2 + 3}{6}$$

$$p_1 \cdot p_2 = K_1 \cdot 6 \Rightarrow K_1 = \frac{p_1 \cdot p_2}{6}$$

$$\lim_{s \to 0} G(s) \stackrel{!}{=} 1$$

$$\Rightarrow \frac{36 \cdot K_{vf}}{6K_1} = 1 \Rightarrow K_{vf} = \frac{K_1}{6}$$

a) $K_1 = 24$; $K_2 = 3{,}5$; $K_{vf} = 4$

b) $K_1 = 48$; $K_2 = 3.5$; $K_{vf} = 8$

c) $K_1 = 266{,}7$; $K_2 = 12{,}8$; $K_{vf} = 44{,}45$

solution optimal control

1 Riccati Equation:

$$s^2 \underbrace{\left(-\frac{1}{r}\cdot b^2\right)}_{a_1} + s\underbrace{(2a)}_{b} + \underbrace{q}_{c} = 0$$

Only positive Solution:

$$s_+ = \frac{-b+\sqrt{b^2-4a_1c}}{2a_1}$$

$$= \frac{-2a+\sqrt{4a^2+4\frac{1}{r}b^2q}}{-2\cdot\frac{1}{r}\cdot b^2}$$

$$= \frac{r}{b^2}\left(a+\sqrt{a^2+\frac{b^2}{r}\cdot q}\right)$$

$$k = \frac{1}{r}\cdot b\cdot s_+ = \frac{1}{b}\left(a+\sqrt{a^2+b^2\cdot\frac{q}{r}}\right)$$

2 $q\to 0$

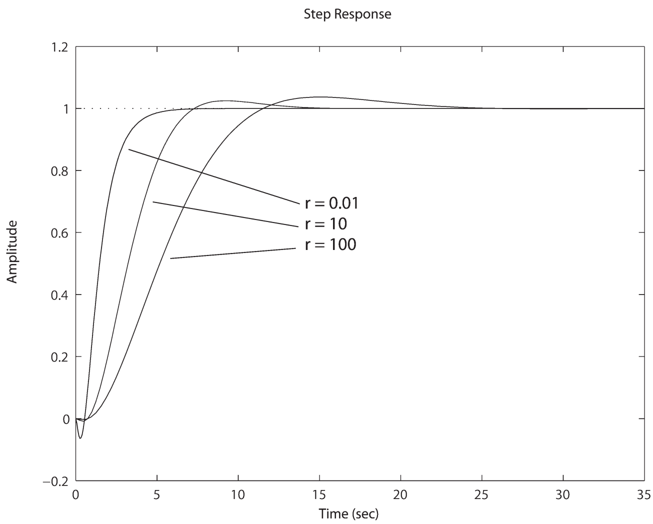
$$k = \begin{cases} 0\,; & \text{if } a<0 \text{ (open loop)} \\ \frac{2a}{b}\,; & \text{if } a\geq 0 \end{cases}$$

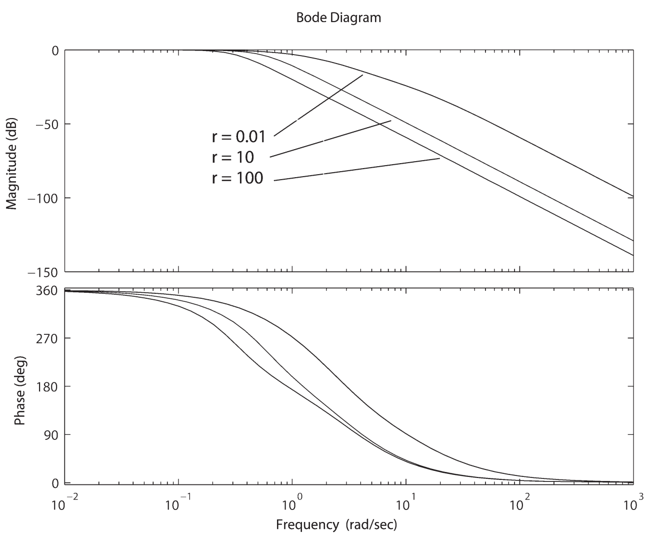
solution LQ-regulator of an inverted pendulum

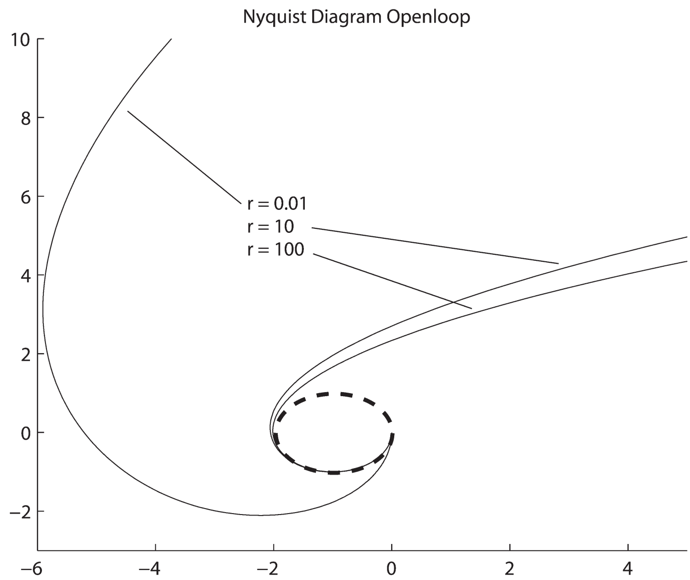
1. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1{,}11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11{,}1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1{,}11 \\ 0 \\ -1{,}11 \end{bmatrix}$; $C = [1\ 0\ 0\ 0]$; $D = 0$

$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ R1 = 0,01
R2 = 10
R3 = 100

```
[K1, S1, E1] = lqr (A, B, Q, R1)
[K2, S2, E2] = lqr (A, B, Q, R2)
[K3, S3, E3] = lqr (A, B, Q, R3)

SYS1 = ss (A - B*K1, B, C, D)
SYS2 = ss (A - B*K2, B, C, D)
SYS3 = ss (A - B*K3, B, C, D)

figure1; holdon
step(K1(1)*SYS1); step (K2(1)*SYS2); step (K3(1)*SYS3)
hold off

figure 2; hold on
bode(K1(1)* SYS1) ; bode (K2(1)* SYS2); bode (K3(1)*SYS3)
hold off

SYS01 = ss (A, B, K1, 0)
SYS02 = ss (A, B, K2, 0)
SYS03 = ss (A, B, K3, 0)
W = logspace(-1, 4, 500)
[Re1, Im1] = nyquist (SYS01, w)
[Re2, Im2] = nyquist (SYS02, w)
[Re3, Im3] = nyquist (SYS03, w)
figure 3; holdon
plot(Re1(:), Im1(:)); plot(Re2(:), Im2(:)); plot (Re3(:), Im3(:))
hold off ; AXIS([-6 5 -3 10])
```


Step Response
Amplitude
Time (sec)
r = 0.01
r = 10
r = 100
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
5
10
15
20
25
30
35


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
r = 0.01
r = 10
r = 100
0
−50
−100
−150
360
270
180
90
0
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}



Discussing the diagram, none of the nyquist curves crosses a circle with radius 1 and center (-1,-i). This means, that the phase margin is at least 60° or (not both together) the gain margin is of 0.5 to ∞, as mentioned in the theory.

state regulator including integral part

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+3}\ ; \quad \underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix} \circ \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \circ u$$

C BB

$$Y = [1 \;\; 0] \circ \underline{x}$$

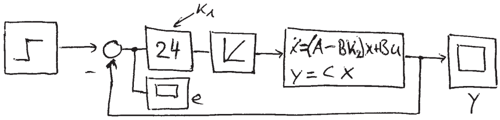
$$\begin{bmatrix} \dot{e} \\ \underline{\dot{m}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -3 \end{bmatrix} \circ \begin{bmatrix} e \\ \underline{m} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \circ u$$

AA A B

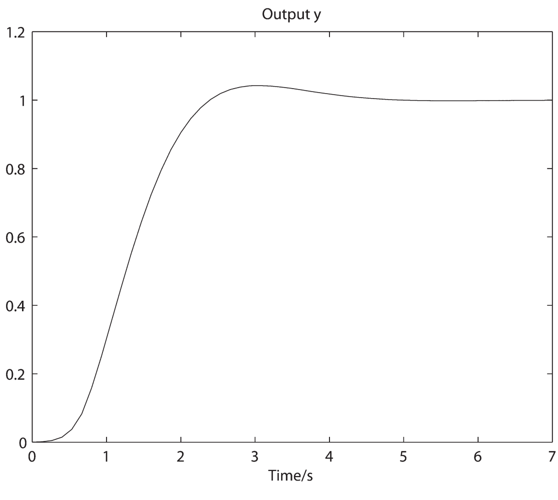
$$P = [-12;\ -1+j,\ -1-j]$$

$$K = place(AA, BB, P)$$

$$\Rightarrow K_1 = 24\ ;\ K_2 = [23 \;\; 11]$$



As discussed in the theory, the steady-state-error is zero.



Chapter 3

State Space regulator with observer

1 Observer

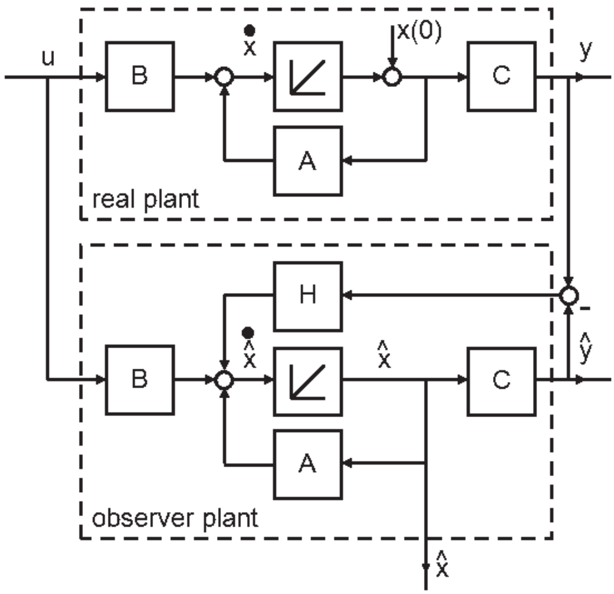


Figure 3.1: observer design

The State Space regulator design assumed that all the state variables are available to realize the feedback. However, for most real existing systems this is not the case. Either it may not be possible due to physical reasons to measure all the state variables as wished or the cost of the sensors may be a limitation. In this chapter it is shown how to calculate all the state variables of a system from at least one or a few measured states. One method of estimating the states is to calculate a model of the plant in the regulator software.

$$\dot{\hat{x}} = A\hat{x} + Bu$$

The estimate of the true state x is $\hat{x}$ in this case. This estimator or observer will work properly if we know the initial condition $x(0)$ and set $\hat{x}(0)$, the estimated initial condition, equal to it. Because the initial condition of the system is not (or not well) known, the error of the estimated state would go to zero too slowly. Note here, that in this case, A, B and u(t) are known. In most real systems, at least the A and B Matrixes are not well known due to parameter uncertainty. This fact would also cause a divergence of the estimated from the true state. To investigate the dynamics of this estimator, we define the error as

$$\tilde{x} = x - \hat{x}$$

and furthermore

$$\dot{\tilde{x}} = A\tilde{x} \quad \text{or} \quad \tilde{x}(0) = x(0) - \hat{x}(0)$$

Here, the error is converging to zero at the same velocity as the dynamics of A. However, there is no possibility to influence the velocity at which it converges to the true state. For a faster convergence, a feedback H is introduced. It is correcting the model with an additional signal, using an amplification of the difference between measured and estimated state(s). The equation for this is shown in figure 3.1

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \quad \text{or} \quad \dot{\hat{x}} = (A - HC)\hat{x} + Bu + Hy$$

The error equation and the characteristic error equation is then

$$\dot{\tilde{x}} = (A - HC)\tilde{x} \quad \text{respectively} \quad det[sI - (A - HC)] = 0$$

The dynamics of the error system can be choosed as much faster than the old dynamics of the system matrix of the plant A. If H is designed in a way, that A-HC has fast and stable eigenvalues then $\tilde{x}$ will converge to zero, indepedant of the input function u(t) and independant of the initial condition $\tilde{x}(0)$. H can typically be designed in a way that the error system is still stable and the error is small enough, even with minor modeling errors. Note that the system behaviour of the plant and the observer are not the same. The plant is a physical system such as a thermal process, an electrical or a mechanical system. The observer is in contradiction a software unit calculating the estimated state. Note again, that in all considerations it is assumed that A,B and C are identical in the real plant and in the observer plant. The dynamics of the error would be different, if there would not be an accurate model of the plant. H can be calculated in exactly the same way as K. It can be found by pole placement and comparision of the coefficients, or as LQR approach. See also the solved exercises.

2 State Space regulator combined with observer

Now an investigation about the influence on the system dynamics using $\hat{x}$ instead of x is needed, if a State Space regulator is combined with an observer. In the following, the closed loop characteristic equation and the open loop compensator transfer function are treated. The basic diagram is showed in figure 3.2. The plant equation with a feedback is

$$\dot{x} = Ax - BK\hat{x}$$

this can be written using the state error $\tilde{x}$

$$\dot{x} = Ax - BK(x - \tilde{x}) \quad \text{or} \quad \dot{x} = (A - BK)x + BK\tilde{x}$$

The dynamics of the whole system is obtained by combining the equations for $\dot{x}$ and $\dot{\tilde{x}}$

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - HC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

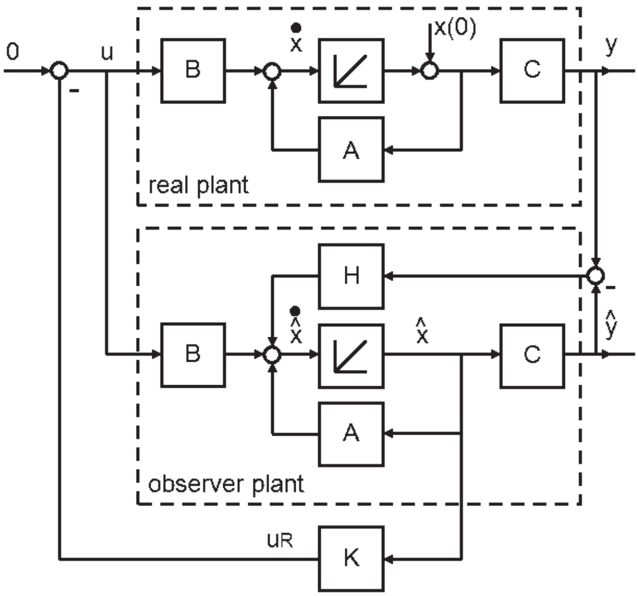


Figure 3.2: State Space regulator with observer design

The characteristic equation (closed loop) is

$$det \begin{pmatrix} sI - A + BK & -BK \\ 0 & sI - A + HC \end{pmatrix} = 0$$

Because the matrix contains a zero block, it can be separated in

$$det(sI - A + BK) \cdot det(sI - A + HC) = 0$$

This means, that the poles of the whole system consist of the control poles as well as of the estimator poles. The effect is nothing else, that the design of the regulator and the observer can be done indepently. The poles of the whole system are an addition of both, the regulator and the observer. This fact is called 'separation principle'. Figure 3.3 shows another block diagram, which is exactly the same as the one in Figure 3.2. The State Space regulator with integrated observer corresponds to a compensator. Note that in this diagram, the error

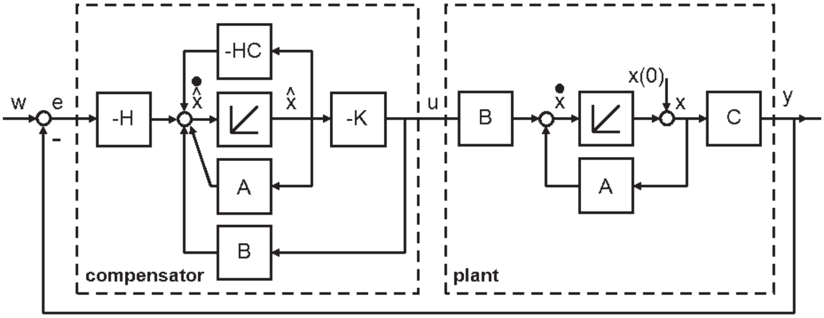


Figure 3.3: State Space regulator with observer as compensator

e is visible again, as it is familar using conventional regulators. The state equation for this compensator is

$$\dot{\hat{x}} = (A - BK - HC)\hat{x} - He \quad \text{and} \quad u = -K\hat{x}$$

and the characteristic equation is

$$det(sI - A + BK + HC) = 0$$

the input-output-description (transfer function) is calculated by

$$C(s) = \frac{U(s)}{E(s)} = K(sI - A + BK + HC)^{-1}H$$

Above, for a LQ regulator a phase margin of at least 60° or a gain margin of 0.5 to ∞ was found. The question is, if a optimal controller with observer has the same properties. The answer, due to the additional singularities from the observer, is NO!

3 Linear Quadratic Gaussian Regulator (LQG) with Loop Transfer Recovery (LTR)

The basic idea of the LQG-LTR-procedure is to determine both, the state controller and the feedback of the observer by a dual design.

Thus, new possibilities for increasing the robustness must be found. Very often the way mentioned below is followed. It gives a solution in a way, that for better loop transfer recovery, the sensitivity for noise is increased. Thus, the better the noise quality of the sensor signals, the better is the loop transfer recovery. The observer-amplification H is calculated by

$$H = (CS)^T$$

where the Matrix S is (as by calculating the LQ-Regulator) the only positive solution of the Riccati equation

$$AS + SA^T - SC^TCS = -(1+\rho)BB^T$$

For minimal phased systems, it can be shown, that the solution of the LQG-LTR-procedure converges by increasing ρ to the frequency response of the LQ-Regulated system. Then, the observer gets faster (and therefore more sensitive to noisy signals). This means due to the separation principle that the observer poles are left far away from the other system poles. In the followong, two approaches of LTR are shown using Matlab.

3.1 LTR using Matlab command ltrsyn

The steps are as follows:

1. Design of LQ-regulator
2. Design the loop transfer recovery using K (from LQ-regulator) and the weighting factor ρ

The result of the ltrsyn-routine is the complete compensator, wich contains K (state regulator) as well as H (observer feedback).

3.2 LTR using a dual LQR design for the State Space regulator K as well as for Observer feedback H

The steps are as follows:

1. Find K: K = lqr(A,B,C'*C,r), C'*C and r are the weighting factors

2. Find H: H’ = lqr (A’,C’,B*B’,q), B*B’ and q are the weighting factors

The result is similar to the ltrsyn command.

4 Solved exercises

design of an observer

A system in State Space description is given.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad , \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Only the state $y = x_1$ can be measured directly.

1. is the system observable?
2. design an observer with a characteristic equation $s^2 + 2\zeta\omega_n s + {\omega_n}^2$ ($\zeta = 1.0$ and $\omega_n = 6$)
3. simulate the response of the error $\tilde{x}$ with initial conditions $\tilde{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

LQG-LTR design with Matlab

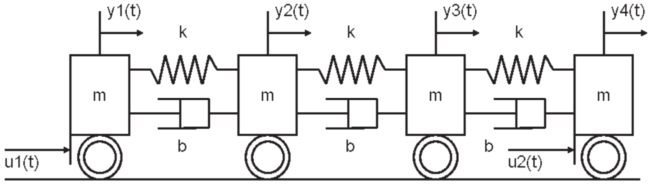
Given is a system

$$A = \begin{pmatrix} -110 & -35 & -8 \\ 32 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 16 \\ 0 \\ 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & 0 & 25 \end{pmatrix} \quad , \quad D = 0$$

1. find a LQR-regulator using $Q = C' * C$ and $R = 1$
2. Plot nyquist diagrams showing the robustness using different RHO's, $XI = 100 * Q$ and $THETA = 1$ (ltrsyn) or q's (dual lqr) and find the observer's feedback H.

four-mass-swinger

Given is the system below, where $u_1(t)$ and $u_2(t)$, the inputs, are forces and y_1, y_2, y_3, y_4, the outputs, are positions. Assume that $m = 1kg$, $k = 36N/m$, $b = 0.6Ns/m$.

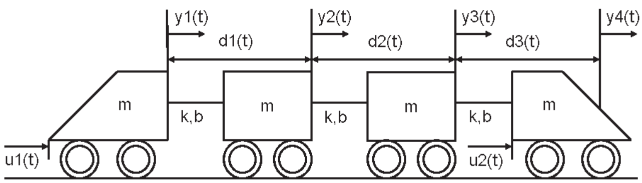


The state variables are $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4$, $x_5 = \dot{y}_1$, $x_6 = \dot{y}_2$, $x_7 = \dot{y}_3$, $x_8 = \dot{y}_4$.

1. Find the State Space description for this system.
2. Design a State Space regulator with closed loop poles $[-2 \pm i \cdot 3, -2 \pm i \cdot 4, -3 \pm i \cdot 3, -3 \pm i \cdot 4]$. Design an observer with poles [-15, -15, -15, -15, -16, -16, -16, -16].
3. Compare the behaviour of the system without and with observer (LQR vs. LQG-LTR). Compare the responses of initial values [-1 -1 -1 -1 0 0 0 0] (Initial states of the observer!).
4. Design the State Space regulator and the observer using optimal control.

train

Given is the system below, where $u_1(t)$ and $u_2(t)$, the inputs, are forces. d_1, d_2 and d_3 are the differences between the positions of the waggons. Assume that $m = 1kg$, $k = 36N/m$, $b = 0.6Ns/m$.



The state variables are $x_1 = d_1$, $x_2 = d_2$, $x_3 = d_3$, $x_4 = \dot{d}_1$, $x_5 = \dot{d}_2$, $x_6 = \dot{d}_3$, $x_7 = \dot{y}_1$.

1. Find the State Space description for this system with one output $\dot{y}_1$.

2. Design a State Space regulator with closed loop poles $[-2 \pm i \cdot 3$, $-2 \pm i \cdot 4$, $-3 \pm i \cdot 3$, -10]. Design an observer with poles [-20, -20, -20, -20, -21, -21, -21].

3. Compare the behaviour of the system without and with observer (LQR vs. LQG-LTR).

4. Plot the step response of the output $\dot{y}_1$. Find a suitable prefilter K_{vf}.

5. Design the State Space regulator and the observer using optimal control.

Solutions

solution design of an observer

1) $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}; \det(Q) = 3$

$\longrightarrow$ observable

2) $\det(sI - (A - HC)) =$

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s-2+h_1 & -3 \\ 1+h_2 & s-4 \end{bmatrix}\right)$$

$$= (s-2+h_1)(s-4) + 3(1+h_2) = s^2 - 2s + h_1 s - 4s + 8 - 4h_1 + 3 + 3h_2$$

$$= s^2 + s(-6+h_1) + 11 - 4h_1 + 3h_2$$

$12 = h_1 - 6 \quad \Rightarrow \quad h_1 = 18$

$36 = 11 - 4h_1 + 3h_2 \Rightarrow h_2 = \dfrac{36 - 11 + 72}{3} = 32{,}33$

$\Rightarrow H = \begin{bmatrix} 18 \\ 32{,}33 \end{bmatrix}$

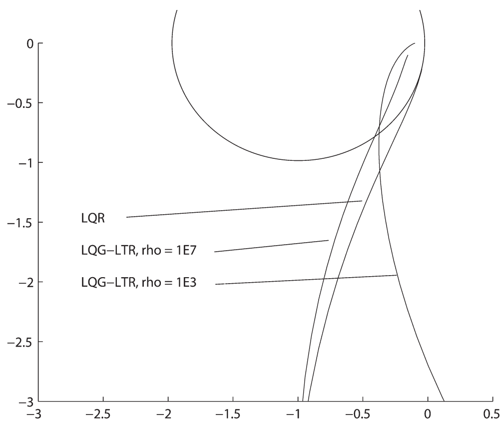
3) $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ C = [1\ 0];\ D = [0];\ H = \begin{bmatrix} 18 \\ 32{,}33 \end{bmatrix}$

```
Ab = A - H*C
C1 = [1 0]; C2 = [0 1]
SYS1 = SS (Ab, B, C1, D); SYS2 = SS(Ab, B, C2, D)
INITIAL (SYS1, [1 -1])
hold on
INITIAL (SYS2, [1 -1])
hold off
```

solution LQG-LTR design with Matlab

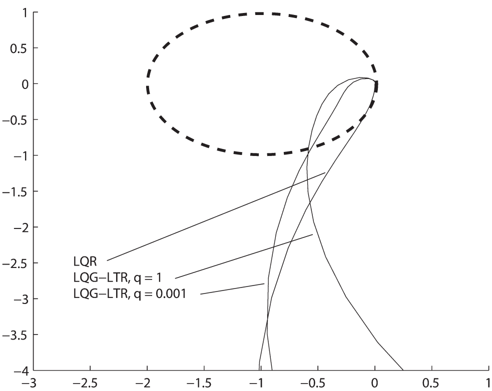
solution using Matlab ltrsyn

```
G= ss (A,B,C,D)
K= lqr (A,B, C'*C,1)
L = ss (A,B,K,0)   % Open Loop
XI =100·C'*C ; THETA =1
RHO =[1E3,1E7]; W= logspace (0,4)
[Re,Im] = nyquist (L,W)
[KA,SVL,W1]= ltrsyn (G,K,XI,THETA,RHO,W)
 clg ; hold on
 plot (SVL(1:1:50,1),SVL(1:1:50,3));
 plot (SVL(1:1:50,2),SVL(1:1:50,4));
 plot (Re(:),Im(:))
 AXIS ([-3 0,5 -3 0])
```



solution using Matlab H' = lqr (A',C',B*B',r) and K = lqr(A,B,C'*C,r)

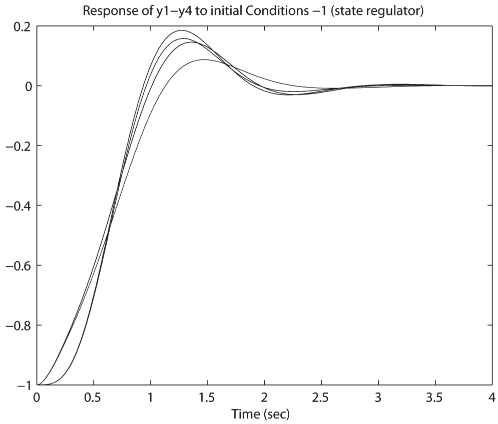
```
G = ss (A, B, C, D)
r = 1 ; q1 = 1 ; q2 = 0.001 ;
K = lqr (A, B, C'*C, r) ;
H1 = lqr (A', C', B*B', q1) ; H1 = H1' ;
H2 = lqr (A', C', B*B', q2) ; H2 = H2' ;
LQR = ss (A, B, K, 0)
LTR1 = ss (A-B*K-H1*C, H1, K, 0) * G ;
LTR2 = ss (A-B*K-H2*C, H2, K, 0) * G ;
W = logspace (0, 4) ;
[Re, Im] = nyquist (LQR, W) ;
[Re1, Im1] = nyquist (LTR1, W) ;
[Re2, Im2] = nyquist (LTR2, W) ;
clg ; holdon
plot (Re(:), Im(:)) ; plot (Re1(:), Im1(:)) ; plot (Re2(:), Im2(:)) ;
AXIS([-3 1 -4 1]) ;
```

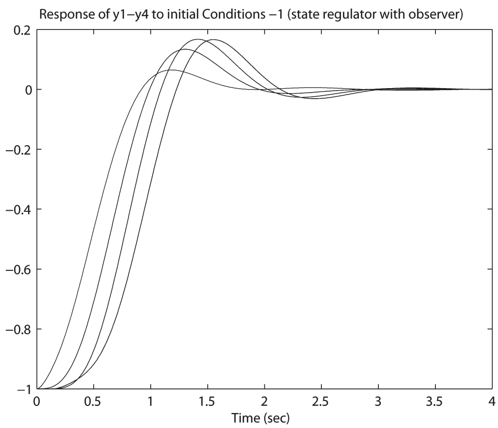


four-mass-swinger

$$m\ddot{y}_1 = k(y_2 - y_1) + b(\dot{y}_2 - \dot{y}_1) + u_1$$
$$m\ddot{y}_2 = -k(y_2 - y_1) - b(\dot{y}_2 - \dot{y}_1) + k(y_3 - y_2) + b(\dot{y}_3 - \dot{y}_2)$$
$$m\ddot{y}_3 = -k(y_3 - y_2) - b(\dot{y}_3 - \dot{y}_2) + k(y_4 - y_3) + b(\dot{y}_4 - \dot{y}_3)$$
$$m\ddot{y}_4 = -k(y_4 - y_3) - b(\dot{y}_4 - \dot{y}_3) + u_2$$

```
A = [0 0 0 0 1 0 0 0;0 0 0 0 0 1 0 0;...
    0 0 0 0 0 0 1 0;0 0 0 0 0 0 0 1;...
    -36 36 0 0 -0.6 0.6 0 0;...
    36 -72 36 0 0.6 -1.2 0.6 0;...
    0 36 -72 36 0 0.6 -1.2 0.6;...
    0 0 36 -36 0 0 0.6 -0.6]
B = [0 0;0 0;0 0;0 0;1 0;0 0;0 0;0 1];

%Position y1-y4
C = [1 0 0 0 0 0 0 0;0 1 0 0 0 0 0 0;0 0 1 0 0 0 0 0;0 0 0 1 0 0
0 0];
D = 0

% State Regulator:
J = [-2+j*3 -2-j*3 -2+j*4 -2-j*4 -3+j*3 -3-j*3 -3+j*4 -3-j*4];
K = place (A,B,J);
%K = lqr(A,B,1*C'*C,[0.0001 0;0 0.0001]);

t = 0:0.01:4;
SYSP = ss(A-B*K,B,C,D);
P = initial (SYSP,[-1;-1;-1;-1;0;0;0;0],t);
figure(1); plot(t,P); axis ([0 4 -1 0.2]);

% Observer:
Ph = [-15 -16 -15 -16 -15 -16 -15 -16];
H = place (A',C',Ph');
% H = lqr(A',C',B*B',0.01* eye(4));
H = H';

AA = [A-B*K B*K; zeros(8,8) A-H*C];
BB = [B;zeros(8,2)];
CC = [C zeros(4,8)];
DD = D;

SYSO = ss(AA,BB,CC,DD)
O = initial (SYSO,[-1;-1;-1;-1;0;0;0;0;0.2;0;0;0;0;0;0;0],t);
figure(2); plot(t,O); axis ([0 4 -1 0.2]);
```


Response of y1−y4 to initial Conditions −1 (state regulator)
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Response of y1−y4 to initial Conditions −1 (state regulator with observer)
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)

train

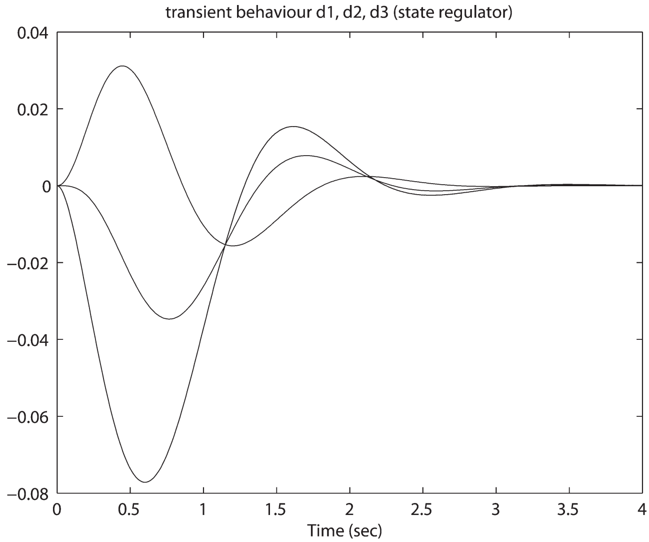
$$\overset{\ddot{d}_1}{\widetilde{m(\ddot{y}_2-\ddot{y}_1)}}=-2kd_1-2b\dot{d}_1+kd_2+b\dot{d}_2-u_1$$

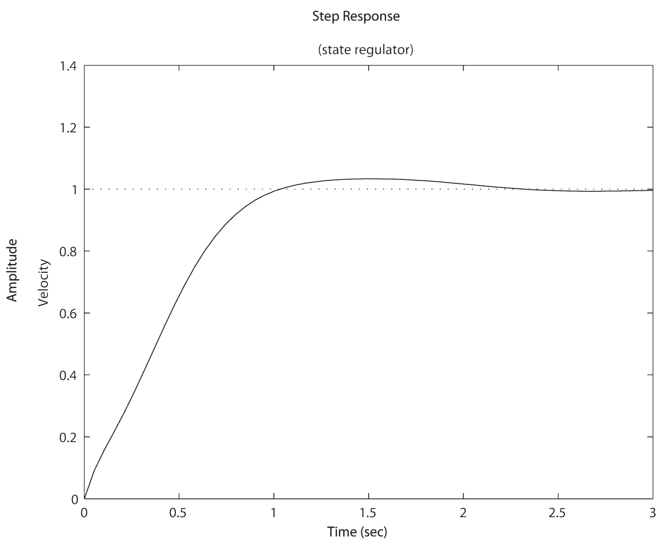
$$\overset{\ddot{d}_2}{\widetilde{m(\ddot{y}_3-\ddot{y}_2)}}=k\,d_1+b\dot{d}_1-2kd_2-2b\dot{d}_2+kd_3+b\dot{d}_3$$

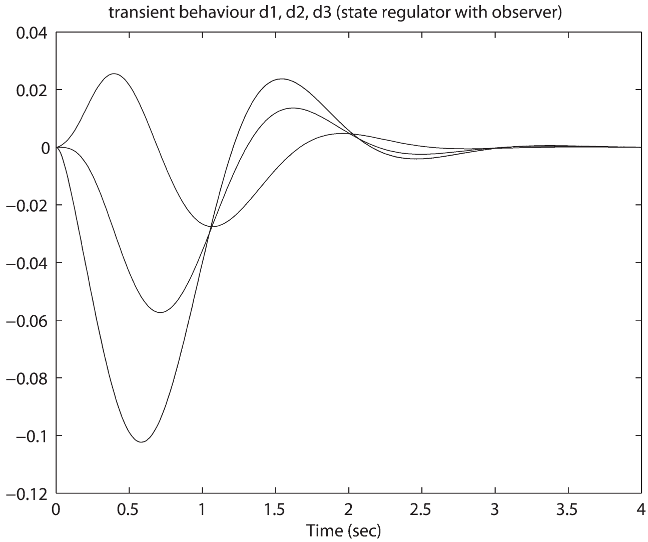
$$\overset{\ddot{d}_3}{\widetilde{m(\ddot{y}_4-\ddot{y}_3)}}=kd_2+b\dot{d}_2-2kd_3-2b\dot{d}_3+u_2$$

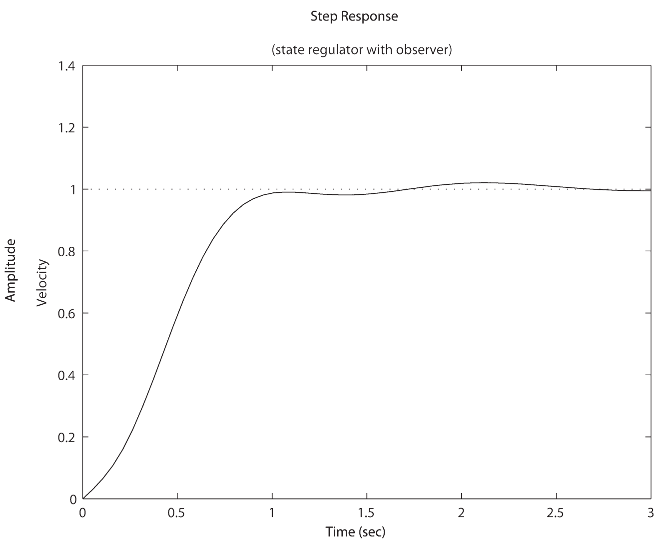
$$m\ddot{y}_1=k\cdot d_1+b\dot{d}_1+u_1$$

```
A = [0 0 0 1 0 0 0;0 0 0 0 1 0 0;0 0 0 0 0 1 0;...
     -72 36 0 -1.2 0.6 0 0;...
     36 -72 36 0.6 -1.2 0.6 0;...
     0 36 -72 0 0.6 -1.2 0;...
     36 0 0 0.6 0 0 0];
B = [0 0;0 0;0 0;-1 0;0 0;0 1;1 0];
C = [0 0 0 0 0 0 1; 1 0 0 0 0 0 0;0 1 0 0 0 0 0;0 0 1 0 0 0 0];
D = zeros (4,2);
```
Output → ↖ Inputs
```
J = [-2+j*3 -2-j*3 -2+j*4 -2-j*4 -3+j*3 -3-j*3 -10];
K = place (A,B,J);
%Q = eye(7);
%K = lqr(A,B,Q,[0.01 0;0 0.01]);

t = 0:0.01:4;
SYSP = ss(A-B*K,B,C,D);
P = initial (SYSP,[0;0;0;0;0;0;-1],t);
[numP,denP] = ss2tf (A-B*K,B,C,D,1);
Kvf = denP (8)/numP(1,8);

figure(1);
plot(t,P(:,2:4))
figure(2);
step(Kvf*numP(1,:),denP);

% with observer:
%H = lqr(A',C',B*B',0.1*eye(4));
%H = H';
I = [-20 -21 -20 -21 -20 -21 -20];
H = place(A',C',I)';

AA = [A-B*K B*K; zeros(7,7) A-H*C];
BB = [B;zeros(7,2)];
CC = [C zeros(4,7)];
DD = zeros (4,2);

SYSP = ss(AA,BB,CC,DD)
O = initial (SYSP,[0;0;0;0;0;0;-1;0.03;0;0;0;0;0;0],t);
[numO,denO] = ss2tf (AA,BB,CC,DD,1);
Kvf = denO (15)/numO(1,15);

figure(3);
plot(t,O(:,2:4));
figure(4);
step(Kvf*numO(1,:),denO);
```


transient behaviour d1, d2, d3 (state regulator)
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Step Response
(state regulator)
Amplitude
Velocity
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Time (sec)


transient behaviour d1, d2, d3 (state regulator with observer)
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.1
−0.12
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Step Response
(state regulator with observer)
Amplitude
Velocity
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Time (sec)

Chapter 4

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