ON THE EMBEDDING OF ZERO-DIMENSIONAL DOUBLE LOOPS IN LOCALLY EUCLIDEAN DOUBLE LOOPS

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ABSTRACT. It is shown that a closed (hence locally compact) zero-dimensional sub-double-loop of a locally Euclidean double loop is always tamely embedded and that its complement is simply connected.

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Throughout this paper we shall consider locally Euclidean double loops $\mathcal{D}$ containing a closed infinite zero-dimensional sub-double-loop $\mathcal{C}$. A locally Euclidean double loop is homeomorphic to some Euclidean space $\mathbb{R}^{2^m}$ with $m \leq 3$ by [6, 7.12] and thus its one-point compactification $\hat{\mathcal{D}}$ is a 2^m -sphere, whereas the one-point compactification $\hat{\mathcal{C}}$ of $\mathcal{C}$ is homeomorphic to the Cantor set (see [3, XI.1.5]). Locally compact connected double loops of dimension at most two are always locally Euclidean by [1] and cannot contain closed zero-dimensional sub-double-loops by [3, XI.9.2]. Hence we shall always assume $\mathcal{D}$ to be four- or eight-dimensional. In these dimensions the existence of closed zero-dimensional sub-double-loops is an open problem. On the other hand, the study of automorphism groups of double loops leads to problems which involve (hypothetically) the existence of closed zero-dimensional sub-double-loops. In this context, one main problem is the nature of the embedding of a closed zero-dimensional sub-double-loop. As indicated above, this question is closely related to the study of embeddings of Cantor sets in spheres. In this paper we prove that a closed zero-dimensional sub-double-loop $\mathcal{C}$ of a locally Euclidean double loop $\mathcal{D}$ is always tamely embedded (see the definition below) in $\mathcal{D}$.

1. Definitions and Auxiliary Results.

The n -sphere will always be considered as the one-point compactification $\mathbb{R}^n \cup \{\infty\}$. Two embeddings $f, g : X \hookrightarrow Y$ between topological spaces X and Y are called *equivalent* iff there is a homeomorphism h of Y onto itself such that $h \circ g = f$. An embedding $f : [0, 1] \hookrightarrow \mathbb{S}_n$ is called *tame* iff f is equivalent to a piecewise linear embedding. Let $C \subset [0, 1]$ denote the Cantor set and let $C' \subseteq C$. An embedding $f : C' \hookrightarrow \mathbb{S}_n$ is called *tame* iff there is a tame embedding $g : [0, 1] \hookrightarrow \mathbb{S}_n$ satisfying $\text{Im } f \subset \text{Im } g$. Let A be a subspace of a topological space X .

A continuous map $F : A \times [0, 1] \rightarrow X$ is called an *isotopy* of A in X iff $F(\cdot, t)$ is an embedding of A into X for every $t \in [0, 1]$ and $F(\cdot, 0) = \text{id}_A$. A continuous map $F : X \times [0, 1] \rightarrow X$ is called an *isotopy* of X iff $F(\cdot, t)$ is a homeomorphism of X onto itself for every $t \in [0, 1]$ and $F(\cdot, 0) = \text{id}_X$. For two subsets A, B of X the set A is said to be *pushed off B instantly* iff there is an isotopy F of X such that $F(A, t) \cap B = \emptyset$ for every $t \neq 0$. The set A is said to be *pushed off B locally* iff for every $x \in B$ there is a neighborhood U of x in B and an isotopy F of X such that $F(A, t) \cap U = \emptyset$ for every parameter $t \neq 0$.

The following two results concerning the existence of certain homeomorphisms of an n -cell or an n -sphere are required later. Note that we consider n -cells always to be closed, i.e. they are homeomorphic to the closed unit cube in $\mathbb{R}^n$.

(1.1) Lemma. *Let B be an n -cell or an n -sphere. Suppose that $B_0, \dots, B_k \subset \text{int } B$ are tamely embedded n -cells, where the sets $B_1, \dots, B_k$ are pairwise disjoint. Define $B' := B \setminus \bigcup_{i=1}^k \text{int } B_i$ and let $K \subseteq \text{int } B_0 \cap \text{int } B'$ and $L \subseteq B \setminus B_0$ be compact sets with $\text{int } K \neq \emptyset$. Let $V \subseteq \text{int } B' \setminus B_0$ be a tamely embedded n -cell in B and let $p \in \text{int } V$. Then there is a homeomorphism $h : B \rightarrow B$ satisfying*

- a) $p \in \text{int } h(K)$, $h(K) \subset \text{int } V$ and $h(L) \cap V = \emptyset$.
- b) $h|_{\partial B} = \text{id}_{\partial B}$.
- c) $h|_{B_i} = \text{id}_{B_i}$ for any $1 \leq i \leq k$.

Proof. Because B_0 is tamely embedded in B and K lies in the intersection $\text{int } B_0 \cap \text{int } B'$, there exists an n -cell X which is a subpolyhedron in B' and contains K in its interior. Being tamely embedded in B , the n -cells X and V are homeomorphic via a piecewise linear map. Since the n -cells $B_1, \dots, B_k$ are tamely embedded and the boundary ∂B is a subpolyhedron of B , theorem 3 of [4] implies that there is a piecewise linear homeomorphism $h : B \rightarrow B$ mapping X onto V satisfying properties b) and c) and $p \in \text{int } h(K)$. In particular, we have $h(K) \subset \text{int } V$ and the sets $h(L)$ and V are disjoint, because $h(B \setminus X) \cap V = \emptyset$.

(1.2) Corollary. *Let B an n -cell. Let $B_1, \dots, B_k \subset \text{int } B$ be pairwise disjoint, tamely embedded n -cells. Set $B' := B \setminus \bigcup_{i=1}^k \text{int } B_i$ and let $p_1, \dots, p_k \in B'$ be elements having pairwise disjoint n -cells $V_1, \dots, V_k$ in B' as neighborhoods. Then there is a homeomorphism $h : B \rightarrow B$ satisfying the following properties:*

- a) $h|_{\partial B} = \text{id}_{\partial B}$.
- b) $p_i \in \text{int } h(B_i)$ and $h(B_i) \subseteq \text{int } V_i$ for any $1 \leq i \leq k$.

Proof. Apply Lemma (1.1) k times.

2. Tameness Criteria.

In this section we shall prove three tameness criteria for Cantor sets being embedded in n -spheres. We start with a tameness criterion that was first formulated in [5] for Cantor sets being embedded in $\mathbb{R}^n$. We shall give a detailed proof of this statement here, because in [5] only a sketch of proof is included.

(2.1) Theorem. *A Cantor set C is tamely embedded in $\mathbb{S}_n$ if and only if for every $\varepsilon > 0$ there is a family $\{C_{\varepsilon,i}\}_{i=1,\dots,k}$ of pairwise disjoint tame n -cells $C_{\varepsilon,i}$ with diameter less than ε such that*

$$C \subset \bigcup_{i=1}^k \text{int } C_{\varepsilon,i}.$$

Proof. (cp. [5]). Let C be tamely embedded in $\mathbb{S}_n$. Then there is a homeomorphism of $\mathbb{S}_n$ which maps C to the unit interval of the x_1 -axis in such a way that the image of C is the ternary Cantor set. The family of sets $\{C_{\varepsilon,i}\}_{i=1,\dots,k}$ can now be written down easily. To prove the converse, inductively choose coverings $\mathcal{C}_k := \{C_{k,i}\}_{i=1,\dots,n_k}$ of C with the following properties:

- (C1) For each $i \in \{1, \dots, n_k\}$ the set $C_{k,i}$ is a tame n -cell.
- (C2) For each $i \neq j$ the sets $C_{k,i}$ and $C_{k,j}$ are disjoint.
- (C3) For each $i \in \{1, \dots, n_k\}$ the diameter of $C_{k,i}$ is less than 2^{-k} .
- (C4) For each $i \in \{1, \dots, n_{k+1}\}$ there is a $j \in \{1, \dots, n_k\}$ such that $C_{k+1,i} \subset \text{int } C_{k,j}$.

Because of properties (C3) and (C4) we have $C \subseteq \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} C_{k,i}$.

For $k \in \mathbb{N}$ set $C_k := \bigcup_{i=1}^{n_k} C_{k,i}$. Let α be a tame arc in $\mathbb{S}_n$ intersecting the interior of all n -cells $C_{1,i}$ for $i \in \{1, \dots, n_1\}$. For $k \geq 2$ and $i \in \{1, \dots, n_k\}$ we define the families

$$\mathcal{D}_{k,i} := \{X \in \mathcal{C}_k ; X \subset \text{int } C_{k+1,i}\} = \{D_{k,i,j}\}_j$$

und set $D_{k,i} := \bigcup \mathcal{D}_{k,i}$.

Step 1. Recursive definition of homeomorphisms $g_k : \mathbb{S}_n \rightarrow \mathbb{S}_n$.

a) $k = 1$.

For each $i \in \{1, \dots, n_1\}$ choose $|\mathcal{D}_{2,i}|$ pairwise distinct points $p_{2,i,j} \in \alpha \cap (\text{int}(C_{1,i}) \setminus D_{2,i})$ and $|\mathcal{D}_{2,i}|$ pairwise distinct neighbourhoods $V_{2,i,j}$ of $p_{2,i,j}$ in $\text{int}(C_{1,i}) \setminus D_{2,i}$ such that $\text{diam } V_{2,i,j} < 2^{-(2+1)}$. We may also assume that the neighborhoods $V_{2,i,j}$ are tame n -cells. Applying Lemma (1.1) we obtain n_1 homeomorphisms $g_{1,i} : C_{1,i} \rightarrow C_{1,i}$ satisfying

$$(H1) \quad g_{1,i}|_{\partial C_{1,i}} = \text{id}_{\partial C_{1,i}}$$

$$(H2) \quad p_{2,i,j} \in \text{int } g_{1,i}(D_{2,i,j}) \text{ and } g_{1,i}(D_{2,i,j}) \subseteq \text{int } V_{2,i,j} \text{ for all } j.$$

Due to property (H1) the homeomorphisms $g_{1,i}$ can be extended to homeomorphisms of the n -sphere via the identity on $\mathbb{S}_n \setminus C_{1,i}$. Define the homeomorphism g_1 as

$$g_1 := g_{1,1} \circ g_{1,2} \circ \dots \circ g_{1,n_1} : \mathbb{S}_n \rightarrow \mathbb{S}_n.$$

Because of property (H2) and the selection of the points $p_{2,i,j}$ we have $\alpha \cap \text{int } g_1(C_{2,i}) \neq \emptyset$ for all $i \in \{1, \dots, n_2\}$.

b) Recursion step from k to $k+1$.

Assume that homeomorphisms g_s of the n -sphere have been defined already for any integer $s \in \{1, \dots, k\}$. Then $h_k := g_k \circ g_{k-1} \circ \dots \circ g_1 : \mathbb{S}_n \rightarrow \mathbb{S}_n$ is a homeomorphism too. Furthermore we may assume that $\alpha \cap h_k(C_{k+1,i}) \neq \emptyset$ for all $i \in \{1, \dots, n_{k+1}\}$. Hence as before we can choose $|\mathcal{D}_{k+2,i}|$ pairwise distinct points $p_{k+2,i,j} \in \alpha \cap h_k(\text{int}(C_{k+1,i}) \setminus D_{k+2,i})$ and $|\mathcal{D}_{k+2,i}|$ pairwise distinct neighbourhoods $V_{k+2,i,j}$ of $p_{k+2,i,j}$ in $h_k(\text{int}(C_{k+1,i}) \setminus D_{k+2,i})$ which are tame n -cells with $\text{diam } V_{k+2,i,j} < 2^{-(k+3)}$. The sets $h_k(D_{k+2,i,j})$ are pairwise disjoint tame n -cells by construction. Thus we can apply Lemma (1.1) to obtain n_{k+1} homeomorphisms $g_{k+1,i} : h_k(C_{k+1,i}) \rightarrow h_k(C_{k+1,i})$ satisfying

$$(H1) \quad g_{k+1,i}|_{\partial h_k(C_{k+1,i})} = \text{id}_{\partial h_k(C_{k+1,i})},$$

$$(H2) \quad p_{k+2,i,j} \in \text{int } g_{k+1,i}(h_k(D_{k+2,i,j})) \text{ and } g_{k+1,i}(h_k(D_{k+2,i,j})) \subseteq \text{int } V_{k+2,i,j} \text{ for all } j.$$

As before, property (H1) enables us to extend the homeomorphisms $g_{k+1,i}$ to homeomorphisms of the n -sphere via the identity on $\mathbb{S}_n \setminus C_{k+1,i}$. Then

$$g_{k+1} := g_{k+1,1} \circ g_{k+1,2} \circ \dots \circ g_{k+1,n_{k+1}} : \mathbb{S}_n \rightarrow \mathbb{S}_n$$

is a homeomorphism of the n -sphere. Furthermore we have $\alpha \cap \text{int } g_{k+1}(h_k(C_{k+2,i})) \neq \emptyset$ for all $i \in \{1, \dots, n_{k+2}\}$ due to property (H2) and the sets $g_{k+1}(h_k(C_{k+2,i}))$ are pairwise disjoint tame n -cells.

Step 2. Properties of the homeomorphisms g_k and h_k .

(P1) For $k \in \mathbb{N}$ and $i \in \{1, \dots, n_k\}$ we have $\text{diam } h_k(C_{k+1,i}) < 2^{-(k+2)}$.

This follows immediately from property (H2) and the inequality $\text{diam } V_{k+1,i,j} < 2^{-(k+2)}$.

(P2) For $k \in \mathbb{N}$ and $x \in \mathbb{S}_n$ we have $d(x, g_k(x)) < 2^{-(k+1)}$.

From the construction of the homeomorphism g_k it follows that g_k moves only points lying in one of the sets $h_{k-1}(C_{k,i})$. Because of $\text{diam } h_{k-1}(C_{k,i}) < 2^{-(k+1)}$, which holds by (P1), the assertion follows.

(P3) For $k \in \mathbb{N}$ and $M \subseteq \mathbb{S}_n$ we have $\text{diam } g_k(M) < \text{diam } M + 2^{-k}$.

For $x, y \in M$ we have

$$d(g_k(x), g_k(y)) \leq d(g_k(x), x) + d(x, y) + d(y, g_k(y)) < 2^{-(k+1)} + \text{diam } M + 2^{-(k+1)},$$

hence $\text{diam } g_k(M) < \text{diam } M + 2^{-k}$.

(P4) The function $h : x \mapsto \lim_{k \rightarrow \infty} h_k(x) : \mathbb{S}_n \rightarrow \mathbb{S}_n$ is welldefined.

We distinguish three cases. If $x \in \mathbb{S}_n \setminus C_1$ we have $g_k(x) = x$ for all $k \in \mathbb{N}$, hence $h(x) = x$. If $x \in C_1 \setminus C_m$ for some $m > 1$, we have $h_k(x) = h_m(x)$ for $k \geq m$ and therefore $h(x) = h_m(x)$. Finally if $x \in C$, we can apply (P2) and obtain

$$\begin{aligned} d(h_k(x), h_{k+m}(x)) &\leq \sum_{i=0}^{m-1} d(h_{k+i}(x), h_{k+i+1}(x)) \\ &\leq \sum_{i=0}^{m-1} d(h_{k+i}(x), g_{k+i+1}(h_{k+i}(x))) \\ &\leq \sum_{i=0}^{m-1} 2^{-(k+i+2)} < 2^{-(k+1)}, \end{aligned}$$

so $(h_k(x))_k$ is a Cauchy sequence. Hence $h(x)$ exists.

(P5) For all $k \in \mathbb{N}$ and $i \in \{1, \dots, n_{k+1}\}$ we have $\text{diam } h(C_{k+1,i}) \leq 2^{-(k-1)}$.

It suffices to show that for $m \geq k$ the relation $\text{diam } h_m(C_{k+1,i}) < 2^{-(k-1)}$ holds. The case $m = k$ is covered by (P1). For $m > k$ we apply (P3) and (P1) and obtain

$$\begin{aligned} h_m(C_{k+1,i}) &< \text{diam } h_{m-1}(C_{k+1,i}) + 2^{-m} \\ &< \text{diam } h_k(C_{k+1,i}) + \sum_{j=1}^{m-k} 2^{-(m-j+1)} \\ &< 2^{-(k+2)} + 2^{-k} \\ &< 2^{-(k-1)}. \end{aligned}$$

(P6) For subsets $A, B \subseteq \mathbb{S}_n$ define $d^*(A, B) := \inf_{y \in B} \sup_{x \in A} d(x, y)$. Then the inequality $d^*(h_k(C_{k+1}), \alpha) < 2^{-(k+2)}$ holds for all $k \in \mathbb{N}$.

For all $k \in \mathbb{N}$ and $i \in \{1, \dots, n_{k+1}\}$ we have $d^*(h_k(C_{k+1,i}), \alpha) < 2^{-(k+2)}$. This follows from (H2) and the relations $\text{diam } V_{k+1,i,j} < 2^{-(k+2)}$ and $\alpha \cap V_{k+1,i,j} \neq \emptyset$. The assertion now follows from the fact that $C_{k+1} = \bigcup_{i=1}^{n_{k+1}} C_{k+1,i}$.

Step 3. The map $h : \mathbb{S}_n \rightarrow \mathbb{S}_n$ is a homeomorphism of the n -sphere satisfying $h(C) \subset \alpha$.

(P7) h is a continuous mapping on $\mathbb{S}_n$.

As we have shown in (P4), it suffices to verify the continuity of h at the elements of the Cantor set C . Let $x \in C$ and let $\mathcal{U} := \{U_j\}_{j \in \mathbb{N}}$ be the maximal subfamily of $\{C_{k,i}\}_{k,i}$ for which $x \in \text{int } U_j$ holds for each $j \in \mathbb{N}$. We may assume the family $\mathcal{U}$ to be indexed in such a way that $U_{j+1} \subset U_j$ for all $j \in \mathbb{N}$. Because of properties (C2) and (C3) we then have $\text{diam } U_j \leq 2^{-j}$, and $\mathcal{U}$ forms a neighborhood base of x . Now let $(x_k)_{k \in \mathbb{N}}$ be a sequence converging to x . We may assume that $x_k \in U_k$. Using (P5) we get the inequality

$$d(h(x), h(x_k)) \leq \text{diam } h(U_k) \leq 2^{-(n-2)},$$

i.e. $\lim_{k \rightarrow \infty} h(x_k) = h(x)$.

(P8) The map h is injective.

Let $x, y \in \mathbb{S}_n$ be distinct elements. If $x, y \in \mathbb{S}_n \setminus C_1$ we have $h(x) = x \neq y = h(y)$. If $x, y \in C_1 \setminus C_m$ for some $m > 1$, we have $h(x) = h_m(x) \neq h_m(y) = h(y)$. Therefore we may assume that $x \in C$. Then there is a set $C_{k,i} \in \mathcal{C}_k$ not containing the element y , but containing the element x in its interior. We distinguish two cases. First, let $y \notin C_k$. Then $h_k(y) \notin h_k(C_k)$ and hence property (H1) guarantees that $h_m(y) = h_k(y)$ holds for every $m \geq k$, and we conclude that $h(y) \notin h_k(C_k)$. For $m > k$ the homeomorphisms g_m leave the sets $h_k(C_k)$ invariant. Thus we have $h(y) \notin h(C_k)$ and hence $h(x) \neq h(y)$. Now let $y \in C_{k,j}$ for some j . Then $i \neq j$ by the choice of $C_{k,i}$ and $h_{k-1}(C_{k,i}) \cap h_{k-1}(C_{k,j}) = \emptyset$ by (H2). In particular we have $h_k(C_{k,i}) \cap h_k(C_{k,j}) = \emptyset$. Because the mappings g_m leave invariant the sets $h_k(C_{k,i})$ and $h_k(C_{k,j})$ for all $m > k$, we infer that $h(C_{k,i}) \cap h(C_{k,j}) = \emptyset$. In particular we have $h(x) \neq h(y)$.

(P9) The map h is surjective.

Assume that $h(\mathbb{S}_n) \neq \mathbb{S}_n$. Since $h(\mathbb{S}_n)$ is compact, there is an element $x \in D := \mathbb{S}_n \setminus h(\mathbb{S}_n)$ with $d(x, h(\mathbb{S}_n)) = u > 0$. Let $k \in \mathbb{N}$ such that $2^{-k} < u$. Setting $y := h_{k-1}^{-1}(x)$ we get

$$\begin{aligned} d(x, h(y)) &\leq \sum_{m \geq k} d(g_m(h_{m-1}(x)), h_{m-1}(x)) \\ &< \sum_{m \geq k} 2^{-(m+1)} \\ &= 2^{-k}. \end{aligned}$$

But this would imply that $h(y) \in D$, a contradiction. For an alternative proof one can use homotopy theory by considering the n -th homotopy group of $\mathbb{S}_n$.

(P10) The map h is a homeomorphism of $\mathbb{S}_n$ onto itself.

This follows immediately from (P7), (P8) and (P9).

(P11) The image $h(C)$ is contained in the tame arc α .

We have $d(h_k(c), \alpha) < 2^{-(k+2)}$ for all $k \in \mathbb{N}$ and any element $c \in C$ by property (P6), since $C \subseteq \bigcap_{k \in \mathbb{N}} C_k$. Hence $d(h(c), \alpha) < 2^{-(k+2)}$ holds for all $k \in \mathbb{N}$, and therefore we have $d(h(c), \alpha) = 0$. Because α is a compact set, this implies $h(c) \in \alpha$ for all $c \in C$.

The next tameness criterion can be found in [7], where it is formulated for Cantor sets being embedded in $\mathbb{R}^n$. To prove this criterion we shall use Theorem (2.1).

(2.2) Theorem. A Cantor set C is tamely embedded in $\mathbb{S}_n$ if and only if for every $\varepsilon > 0$ and every $c \in C$ there is a tame n -cell $B \subset \mathbb{S}_n$ with diameter less than ε such that $c \in \text{int } B$ and $\partial B \cap C = \emptyset$.

Proof (cp. [7]). To prove the assertion, we shall use Theorem (2.1). Let $\varepsilon > 0$. By (2.1) for each $x \in C$ there exists a tame n -cell B_x in $\mathbb{S}_n$ of diameter less than ε such that $x \in \text{int } B_x$ and $\partial B_x \cap C = \emptyset$. Since C is compact, there is a finite subcollection $\{B_1, \dots, B_m\}$ of these n -cells such that their interiors cover C . Define a partition $\{C_1, \dots, C_m\}$ of the Cantor set C by $C_1 := C \cap B_1$ and $C_i := (C \cap B_i) \setminus \bigcup_{j=1}^{i-1} B_j$ for $i > 1$. Because C does not meet the boundary of any of the n -cells B_i , the sets C_i are closed and open in C . Set $B_{1,1} := B_1$. Assume that for $0 < k < m$ we already have defined pairwise disjoint tame n -cells $B_{k,1}, \dots, B_{k,l_k}$ in B_k such that their boundaries miss C . If the set C_{k+1} is void there is nothing left to do, so we may assume C_{k+1} to be non-empty. Applying Theorem (2.1) for each $x \in C_{k+1}$, we get tame n -cells B_x^* in $\text{int } B_{k+1} \setminus \bigcup_{j=1}^k B_j$ whose boundaries miss C . Because the set C_{k+1} is compact, there is a finite subcollection $\{B_{k+1,1}^*, \dots, B_{k+1,l_{k+1}}^*\}$ of the covering $\{B_x^*\}_{x \in C_{k+1}}$ such that the union of their interiors still cover C_{k+1} . For each $i \in \{1, \dots, l_{k+1}\}$ let $C_{k+1,i} := (C_{k+1} \cap \text{int } B_{k+1,i}^*) \setminus \bigcup_{j=1}^{i-1} B_{k+1,j}^*$. The sets $C_{k+1,i}$ are again open and closed sets with respect to C . Now choose pairwise disjoint tame n -cells $B'_{k+1,2}, \dots, B'_{k+1,l_{k+1}}$ in $\text{int } B_{k+1} \setminus \bigcup_{j=1}^{l_{k+1}} B_{k+1,j}^*$. Set $h_{k+1,1} := \text{id}_{B_{k+1}}$ and $B_{k+1,1} := B_{k+1,1}^*$. Applying Lemma (1.1) we inductively obtain homeomorphisms $h_{k+1,i} : B_{k+1} \rightarrow B_{k+1}$ ($i \geq 2$) satisfying the properties (1) to (4) listed below.

- (1) $h_{k+1,i}|_{\partial B_{k+1}} = \text{id}_{\partial B_{k+1}},$
- (2) $h_{k+1,i}(C_{k+1,i}) \subseteq \text{int } B'_{k+1,i},$
- (3) $h_{k+1,i}(C_{k+1} \setminus C_{k+1,i}) \cap B'_{k+1,i} = \emptyset,$
- (4) $h_{k+1,i}|_{B_{k+1,j}^*} = \text{id}_{B_{k+1,j}^*}$ if $j < i$.

Note that as before we may assume the sets $C_{k+1,1}, \dots, C_{k+1,l_{k+1}}$ to be non-void. Because of property (1) the homeomorphisms $h_{k+1,i}$ can be extended to homeomorphisms of the

n -sphere via the identity. Therefore the n -cells $B_{k+1,i} := h_{k+1,i}^{-1}(B'_{k+1,i})$ are tamely embedded in $\mathbb{S}_n$. Properties (2), (3) and (4) assure the n -cells $B_{k+1,i}$ to be pairwise disjoint, their boundaries not meeting C . Noting that $C_{k+1} = \bigcup_{i=1}^{l_{k+1}} C_{k+1,i}$ we derive from property (2) the inclusion $C_{k+1} \subset \bigcup_{i=1}^{l_{k+1}} \text{int } B_{k+1,i}$. The diameters of the n -cells $B_{k+1,i}$ are less than ε , since these n -cells are subsets of B_{k+1} . Altogether we have $C \subset \bigcup_{k=1}^m \bigcup_{i=1}^{l_k} \text{int } B_{k+1,i}$ and the assertion now follows from Theorem (2.1).

Our last tameness criterion will turn out to be most useful in the theory of locally Euclidean double loops. This criterion can be regarded as a 'local version' of a result found in [7].

(2.3) Theorem. *A Cantor set C is tamely embedded in $\mathbb{S}_n$ if and only if C can be pushed off itself locally.*

Proof (cp. [7]). If C is tamely embedded in $\mathbb{S}_n = \mathbb{R}^n \cup \{\infty\}$ we may assume that C lies entirely in the unit interval of the x_1 -axis of $\mathbb{R}^n$. Now define $h : \mathbb{S}_n \times [0, 1] \rightarrow \mathbb{S}_n$ by $h(x_1, \dots, x_n) := (x_1 + t, x_2, \dots, x_n)$ and $h(\infty) = \infty$ which is the desired isotopy of $\mathbb{S}_n$. Conversely let $c \in C$. By hypothesis there is a neighborhood $U_c \subset C$ of c and an isotopy $h : \mathbb{S}_n \times [0, 1] \rightarrow \mathbb{S}_n$ of the n -sphere satisfying $h(C, t) \cap U_c = \emptyset$ whenever $t > 0$. Choose an n -cell B^* of diameter less than ε containing c in its interior, missing $h(C, 1)$, and satisfying the relation $B^* \cap C \subset U_c$. Select another n -cell $B' \subset \text{int } B^*$ which is tamely embedded in $\mathbb{S}_n$ and which contains the element c in its interior. Let V be a closed neighborhood of ∂B^* in B^* that misses the n -cell B' . Furthermore select an open and closed set C' in C lying in $\text{int } B'$ as well as a set $C'' \subset V$ which is open and closed in $C \cap B^*$. Because the set $C \setminus (C' \cup C'')$ is contained in the interior of B^* , we may assume that $h(C \setminus (C' \cup C''), t)$ is contained in $\text{int } B^*$ for all $t \in [0, 1]$. Set $K := (B^* \cap C) \cup \partial B^*$ and $K' := K \setminus (C' \cup C'' \cup \partial B^*)$. Define an isotopy $f : K \times [0, 1] \rightarrow B^*$ by

$$f(x, t) = \begin{cases} x & \text{if } x \in K' \\ h(x, t) & \text{otherwise} \end{cases},$$

and write $f(x, t)$ as $f_t(x)$. Now the map $f_t : K \rightarrow K$ is injective for each $t \in [0, 1]$, since $h(K', t) \cap (C \cup \partial B^*) = \emptyset$. Furthermore each map f_t is an embedding, because the sets K' and $K \setminus K'$ are open in K , and $f_0 = \text{id}_K$ holds by definition. So f is an isotopy of K in B^* . Our goal is to extend f to an isotopy F of B^* by applying [2, Corollary 1.2]. For this we have to show that f can be extended to an isotopy f^* of an appropriate neighborhood of K in B^* . Because $K \setminus K'$ is compact and h is continuous, we infer that $T := h(K \setminus K', [0, 1])$ is a compact set which is disjoint from K' , because h pushes C off itself locally and $h(C \setminus (C' \cup C''), t)$ is contained in $\text{int } B^*$. Being a compact topological space, the n -cell B^* contains disjoint compact neighborhoods V of K' and W of $K \setminus K'$ such

that $h(W, [0, 1]) \cap V = \emptyset$. Furthermore the neighborhood W can be chosen in such a way that $h(W, [0, 1]) \subseteq B^*$. Now we may define the extension $f^* : \text{int}(V) \cup \text{int}(W) \times [0, 1] \rightarrow B^*$ by

$$f^*(x, t) = \begin{cases} x & \text{if } x \in \text{int}(V) \\ h(x, t) & \text{if } x \in \text{int}(W) \end{cases}.$$

Using the same arguments as before we see that f^* is an isotopy of $\text{int}(V) \cup \text{int}(W)$ in B^* which extends f . Applying Corollary 1.2 of [2] we obtain an isotopy F of B^* which is also an extension of f , i.e. $F_t|_{K'} = \text{id}_{K'}$ and $F_t|_{K \setminus K'} = h_t|_{K \setminus K'}$. In particular we have $F_t|_{\partial B^*} = \text{id}_{\partial B^*}$, so F_t can be extended to a homeomorphism of the n -sphere via the identity. We finish the proof by showing that $B := F_1^{-1}(B')$ is an n -cell satisfying the hypothesis of Theorem (2.2). First we have $B \cap (C \setminus C') = F_1^{-1}(B') \cap F_1^{-1}(h_1(C \setminus C')) = \emptyset$, because B' misses $h_1(C \setminus C')$. In addition $C' = F_1^{-1}(C') \subseteq F_1^{-1}(\text{int } B') = \text{int } B$ holds, which implies that $C \cap \partial B = \emptyset$. Secondly, B is a tame n -cell, because B' is a tame n -cell. Finally the diameter of B is less than ε , because B is contained in B^* . Thus C is tamely embedded in $\mathbb{S}_n$ by Theorem (2.2).

3. Tameness of zero-dimensional double loops.

In this final section we shall apply the tameness criteria proved in the last section to a closed zero-dimensional sub-double-loop $\mathcal{C}$ of a locally Euclidean double loop $\mathcal{D}$. We call $\mathcal{C}$ *tamely embedded* in $\mathcal{D}$ if and only if the Cantor set $\widehat{\mathcal{C}}$ is tamely embedded in the sphere $\widehat{\mathcal{D}}$.

(3.1) Theorem. *Let $\mathcal{C}$ be a closed zero-dimensional sub-double-loop of a locally Euclidean double loop $\mathcal{D}$. Then $\mathcal{C}$ is tamely embedded in $\mathcal{D}$.*

Proof. Without loss of generality we may assume that $\mathcal{C}$ is infinite and hence is homeomorphic to a Cantor set minus one point by [3, XI.1.5]. Let $b \in \mathcal{D} \setminus \mathcal{C}$. Because $\mathcal{D}$ is pathwise connected, there is a path γ' between the neutral element 0 of $\mathcal{D}$ and the element b . Since $\mathcal{C}$ is a closed zero-dimensional subset of $\mathcal{D}$, there is an element $a \in \mathcal{C}$ such that the restricted path γ of γ' starting from a and ending in b meets $\mathcal{C}$ only in the point a . Define isotopies F and F' of $\widehat{\mathcal{D}}$ by

$$F(x, t) = x + (\gamma(t) - a)$$

and

$$F'(x, t) = 1/F(1/x, t) = 1/(1/x + (\gamma(t) - a)).$$

These functions are welldefined isotopies, because addition extends continuously to $\widehat{\mathcal{D}} \times \widehat{\mathcal{D}}$ and the left inverse mapping $x \mapsto 1/x$ extends continuously to $\widehat{\mathcal{D}}$ by [3, XI.8.3]. Furthermore we have $F(\mathcal{C}, t) \cap \mathcal{C} = \emptyset$ and $F'(\widehat{\mathcal{C}} \setminus \{0\}, t) \cap \widehat{\mathcal{C}} \setminus \{0\} = \emptyset$. Hence $\widehat{\mathcal{C}}$ can be pushed off itself locally and the assertion follows from Theorem (2.3).

In the study of automorphism groups of locally compact double loops the following result turns out to be quite useful.

(3.2) Corollary. *The complement of a closed zero-dimensional sub-double-loop $\mathcal{C}$ in a locally Euclidean double loop $\mathcal{D}$ is simply connected.*

Proof. Let ω be a closed curve in $\mathcal{D} \setminus \mathcal{C}$. By Theorem (2.1) there is a finite collection of pairwise disjoint tame n -cells B_i in $\widehat{\mathcal{D}}$ such that $\widehat{\mathcal{C}} \subset \text{int } B$ and ω is contained in the complement $\widehat{\mathcal{D}} \setminus B$, where B is the union of the sets B_i . This is trivially true if $\mathcal{C}$ is finite. The complement $\widehat{\mathcal{D}} \setminus B$ is simply connected, since the dimension of $\widehat{\mathcal{D}}$ is at least four and the n -cells B_i are tamely embedded in $\widehat{\mathcal{D}}$. Thus the path ω is null-homotopic in $\widehat{\mathcal{D}} \setminus \widehat{\mathcal{C}} = \mathcal{D} \setminus \mathcal{C}$, which proves that $\mathcal{D} \setminus \mathcal{C}$ is simply connected.

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