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16-Dimensional Smooth Projective Planes with Large Collineation Groups

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Abstract. Smooth projective planes are projective planes defined on smooth manifolds (i.e. the set of points and the set of lines are smooth manifolds) such that the geometric operations of join and intersection are smooth. A systematic study of such planes and of their collineation groups can be found in previous works of the author. We prove in this paper that a 16-dimensional smooth projective plane which admits a collineation group of dimension $d \geqslant 39$ is isomorphic to the octonion projective plane $\mathcal{P}_2\mathbb{O}$. For topological compact projective planes this is true if $d \geqslant 41$. Note that there are nonclassical topological planes with a collineation group of dimension 40.

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1. Introduction

The theory of compact projective planes is presented in the recent book of Salzmann et al. [29]. A main theme of this theory is the classification of *sufficiently homogeneous* compact planes, i.e. planes that admit a collineation group of sufficiently large dimension. For 16-dimensional compact projective planes, we have the following theorem ([29], 85.16):

THEOREM. *Let $\mathcal{P}$ be a* 16*-dimensional compact projective plane. If* $\dim \operatorname{Aut}\mathcal{P} > 40$, *then $\mathcal{P}$ is isomorphic to the Cayley projective plane $\mathcal{P}_2\mathbb{O}$ and* $\operatorname{Aut}\mathcal{P} \cong \mathrm{E}_{6(-26)}$.

Note that the dimension bound of 40 is sharp, since there exist nonclassical compact planes with a 40-dimensional (Lie) group of collineations, [29], 82.27. In this paper, we will study *smooth* projective planes and prove a similar result (see the Main Theorem at the end of this section).

DEFINITION 1.1. A projective plane $\mathcal{P} = (P, \mathcal{L})$ is called *smooth* if the set P of points and the set $\mathcal{L}$ of lines are smooth ($= \mathrm{C}^\infty$) manifolds such that the geometric operations $\vee$ of join and $\wedge$ of intersection are smooth mappings.

It is convenient to identify every line $L \in \mathcal{L}$ with the set of points incident with L. The set $\mathcal{L}_p$ of all lines through some point p is called a *line pencil*. The integer $n = \dim P$ which is called the *dimension* of the projective plane $\mathcal{P}$ is always a power 2^k, where $k = 1, 2, 3, 4$, see [29], Section 54. Moreover, for $n = 2l$ we have $\dim L = \dim \mathcal{L}_p = l$ for any line L and any line pencil $\mathcal{L}_p$.

By [1], for any point $p \in P$ the tangent space $\mathrm{T}_p P$ together with the *tangent spread* $\mathcal{S}_p = \{\mathrm{T}_p K \mid K \in \mathcal{L}_p\}$ induced by the line pencil $\mathcal{L}_p$ forms a locally compact affine translation plane $\mathcal{A}_p$. These affine planes are called *tangent translation planes*. Their projective closures are denoted by $\mathcal{P}_p$ and we put L_∞ as the line at infinity (which is also the translation line). We denote by $\mathcal{A}_2\mathbb{F}$ the classical affine translation plane over the division ring $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and we put $\mathcal{P}_2\mathbb{F}$ as the projective closure of $\mathcal{A}_2\mathbb{F}$. According to [2], (2.3), every continuous collineation of a smooth projective plane $\mathcal{P}$ is in fact smooth. This enables us to compare the results in the topological situation with the results in the smooth case, see the remark after the Main Theorem. The group Γ of continuous (or, equivalently, of smooth) collineations γ of $\mathcal{P}$ is a Lie transformation group (with respect to the compact-open topology) on both the set P of points and the set $\mathcal{L}$ of lines, see [2], (2.4). In particular, any collineation γ is a diffeomorphism of P onto P and of $\mathcal{L}$ onto $\mathcal{L}$. The stabilizer Γ_p of some point $p \in P$ induces an action on the tangent translation plane $\mathcal{A}_p$ via the *derivation mapping*

$$D_p \colon \Gamma_p \to \Sigma_o := \mathrm{Aut}(\mathcal{A}_p)_o \leqslant \mathrm{GL}(\mathrm{T}_p P) \colon \gamma \mapsto \mathrm{D}\gamma(p),$$

where $\mathrm{Aut}\,(\mathcal{A}_p)_o$ is the stabilizer of $\mathrm{Aut}\,(\mathcal{A}_p)$ at the origin o. By [2], (3.3) and (3.9), the map D_p is a continuous homomorphism and $\ker \mathrm{D}_p = \Gamma_{[p,p]}$ is the subgroup of all elations of Γ having p as their center. If a group G acts on a set X, we denote by G_X the kernel of this action and we put $G^X := G/G_X$. For $x \in X$ the stabilizer of x in G is abbreviated by G_x. For a subset Y of X we set $G_Y = \{g \in G \mid \forall y \in Y \colon y^g = y\}$. The connected component of the identity of a topological group G is written as G^1.

Our aim is to prove the following result.

MAIN THEOREM. *Let $\mathcal{P}$ be a* 16*-dimensional smooth projective plane which has a locally compact collineation group Δ of dimension at least* 39. *Then $\mathcal{P}$ is isomorphic (as a smooth projective plane) to the classical Moufang plane $\mathcal{P}_2\mathbb{O}$.*

Remark. There are nonclassical *topological* 16-dimensional projective planes admitting a 40-dimensional group of collineations, see [29], 82.26–82.29. Such planes are always translation planes. It is not known, whether there exist nontranslation planes with a collineation group of dimension 39. Compared to the theorem given at the beginning of this paper the Main Theorem shows that nonclassical smooth 16-dimensional planes are less homogeneous than in the topological situation.

2. Auxiliary Results

Throughout, let $\mathcal{P} = (P, \mathcal{L})$ be a smooth 16-dimensional projective plane with a closed connected subgroup $\Delta \leqslant \Gamma$. We shall always assume that $39 \leqslant \dim \Delta \leqslant 40$. We put $F_\Delta := \{x \in P \cup \mathcal{L} | \forall \delta \in \Delta : x^\delta = x\}$ as the set of fixed elements of Δ.

Very often we implicitely shall make use of Halder's dimension formula for locally compact groups, [7], which is formulated below for convenience.

2.1. HALDER'S DIMENSION FORMULA

If a locally compact Lindelöf group G acts on a separable metric space M, then $\dim G = \dim G_a + \dim a^G$ *for every point $a \in M$.*

Note that dim denotes the covering dimension. Another useful dimension formula is proved in [3], Lemma (1.2).

LEMMA 2.2. *If Δ is a locally compact connected collineation group of a smooth projective plane $\mathcal{P}$ which fixes some point $p \in P$, then* $\dim \Delta = \dim \ker \mathrm{D}_p + \dim \mathrm{D}_p \Delta$ *and* $\dim \mathrm{D}_p \Delta \leqslant \dim \mathrm{Aut}(\mathcal{A}_p)_o$, *where* $\mathrm{D}_p : \Delta \to \mathrm{Aut}(\mathcal{A}_p)_o$ *is the derivation map and* $\mathrm{Aut}(\mathcal{A}_p)_o$ *is the stabilizer of* $\mathrm{Aut}(\mathcal{A}_p)$ *at the origin o. Moreover, we have* $\ker \mathrm{D}_p = \Delta_{[p,p]}$.

Tangent translation planes of $\mathcal{P}$. For the investigation of the collineation group Δ we utilize results of H. Hähl, see [10], (4.2). Let $\mathcal{A} = (A, \mathcal{G})$ be a locally compact affine translation plane of dimension $n = 2l$. We choose some point o in A as well as three distinct lines $W, S, X \in \mathcal{G}_o$ through o. Fixing a 'unit point' e in $X \setminus \{o\}$, the affine translation plane $\mathcal{A}$ is coordinatized by some quasifield Q whose additive group $(Q, +)$ is isomorphic to $\mathbb{R}^l$. Hence, the kernel of the quasifield Q contains the real numbers as a subfield. In particular, the group $(Q, +)$ can be viewed as an l-dimensional real vector space. In this setting, the set A of points can be written as $A = Q \times Q \cong \mathbb{R}^{2l}$, the origin o has coordinates $(0, 0)$, and we have $W = Q \times \{0\}$, $S = \{0\} \times Q$, and $X = \mathrm{diag}(Q \times Q)$. The automorphism group Σ of $\mathcal{A}$ is a semi-direct product $\Sigma = \Sigma_o \ltimes \mathrm{T}$, where $\mathrm{T} \cong \mathbb{R}^{2l}$ is the group of translations and Σ_o is the stabilizer of the origin o. Moreover, the stabilizer $\Sigma_{W,S}$ can be expressed in terms of $\mathbb{R}$-linear mappings of the real vector space Q, namely

$$\Sigma_{W,S} \leqslant \{(B, C) : Q^2 \to Q^2 : (x, y) \mapsto (Bx, Cy) \mid B, C \in \mathrm{GL}(Q)\}.$$

Since we have $X = \mathrm{diag}(Q \times Q)$, the stabilizer of the three lines S, W and X can be written as

$$\Sigma_{W,S,X} \leqslant \{(B, B) : Q^2 \to Q^2 : (x, y) \mapsto (Bx, By) \mid B \in \mathrm{GL}(Q)\}.$$

Now we can formulate a theorem by Hähl (see [8], 2.1 or [29], 81.8), which turns out to be a very effective tool in our proofs.

THEOREM 2.3. *Let $\mathcal{A}$ be an affine locally compact translation plane of dimension $n = 2l$ and let W, S and X be three different lines of $\mathcal{A}$ through the origin o. Let Σ_o^1 be the connected component of the stabilizer Σ_o.*

(a) *The group*

$$\mathsf{M} := \{(A, B) \in \Sigma_o^1 \mid |\det A| = |\det B| = 1\} \leqslant \mathrm{SO}\, l\mathbb{R} \times \mathrm{SO}\, l\mathbb{R}$$

is the largest compact subgroup of the stabilizer $\Omega_2 := (\Sigma^1)_{W,S}$ *and* $\dim \Omega_2/\mathsf{M} \leqslant 2$. *If* $\dim \Omega_2/\mathsf{M} = 2$, *then there is a closed noncompact one-parameter subgroup* P *of* Ω_2 *such that* Ω_2^1 *can be decomposed as* $\Omega_2^1 = \mathsf{M} \cdot \mathsf{P} \cdot \Sigma_{[o,L_\infty]}$.

(b) *The group*

$$\mathsf{N} := \{(A, A) \in \Sigma_o^1 \mid |\det|, A| = 1\} \leqslant \mathrm{SO}_l\mathbb{R}$$

is the largest compact subgroup of the stabilizer $\Omega_3 := (\Sigma^1)_{W,S,X}$ *and* $\dim \Omega_3/\mathsf{N} \leqslant 1$. *If* $\dim \Omega_3/\mathsf{N} = 1$, *then there is a closed noncompact one-parameter subgroup* P *of* Ω_3 *such that* $\Omega_3^1 = \mathsf{N} \times \mathsf{P}$.

We will also use the following strong result of J. Otte, [20] and [21], on smooth translation planes.

OTTE'S THEOREM. 2.4. *Every smooth projective translation plane is isomorphic (as a smooth projective plane) to one of the classical projective planes $\mathcal{P}_\mathbb{F}$ defined over an alternative field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

Suppose now that Δ has some fixed flag (p, L). We are going to determine the tangent translation plane $\mathcal{A}_p = (\mathrm{T}_p P, \mathcal{S}_p)$ of $\mathcal{P}$ at the fixed point p. In order to do that, we consider the derivation map

$$\mathrm{D}_p \colon \Delta \to \mathrm{Aut}(\mathcal{A}_p)_{o, T_p L} \colon \delta \mapsto \mathrm{D}_p \delta,$$

where o is the origin of the point set $\mathrm{T}_p P$. Note that $\mathrm{D}_p\Delta$ fixes the subspace $\mathrm{T}_p L$, because Δ fixes L. Since a smooth projective translation plane is classical by Otte's Theorem, we may assume that $\dim \Delta_{[p,p]} \leqslant 15$. Because $\mathcal{P}_p$ is a compact 16-dimensional translation plane, we obtain from Lemma (2.2)

$$\begin{aligned} \dim \Sigma &= \dim \Sigma_o + 16 \geqslant \dim \mathrm{D}_p\Delta + 16 \\ &= \dim \Delta - \dim \Delta_{[p,p]} + 16 \geqslant \dim \Delta + 1 \geqslant 40. \end{aligned}$$

The compact translation planes with a collineation group of dimension at least 40 are completely classified. This classification is due to H. Hähl, see [12], [13], and compare [29], 82.26(5), 82.27, and 82.21. The following theorem collects the relevant information.

THEOREM 2.5. *Let $\mathcal{A}$ be a topological* 16*-dimensional compact projective translation plane with collineation group* A*. Assume that* $\dim \mathsf{A} \geqslant 40$*. Then the following assertions hold.*

(i) *Either* $\mathcal{A} \cong \mathcal{P}_2\mathbb{O}$ *and* $\dim \mathsf{A} = 78$*, or* $\mathcal{A}$ *is isomorphic to some projective plane defined over the mutations* $\mathbb{O}_\alpha$ *(see* [6]*, XI.14 or* [29], 82.27) *and* $\dim \mathsf{A} = 40$.

(ii) *Let* L_∞ *be a translation axis of* $\mathcal{A}$ *and let* R, S *be two lines such that their intersection* $o = R \wedge S$ *is not incident with* L_∞*. Let* B *denote the connected component of the stabilizer* $\mathsf{A}_{R,S,L_\infty}$*. Then* B *contains a largest compact subgroup* M, $\mathsf{B} = \mathsf{M} \cdot \mathsf{P} \cdot \mathsf{B}_{[o,L_\infty]}$ *for some closed noncompact one-parameter subgroup* P *of* B*, and* $\mathsf{B}_{[o,L_\infty]} \cong \mathbb{R}_{>0}$*. If* $\mathcal{A} \cong \mathcal{P}_2\mathbb{O}$ *then* $\mathsf{M} \cong \mathrm{Spin}_8\mathbb{R}$*, and we have* $\mathsf{M} \cong \mathrm{G}_{2(-14)}$ *in the case of* $\dim \mathsf{A} = 40$.

The dimension of the stabilizer of two lines. Let $K \neq L$ be another line of $\mathcal{P}$ through the point p. We denote by Φ the connected component of the stabilizer $\Gamma_{K,L}$ and by Ψ a Levi subgroup of Φ. We are going to consider the derivation map $\mathrm{D}_p \colon \Phi \to \Omega$, where Ω is the connected component of the stabilizer $\mathrm{Aut}(\mathcal{A}_p)_{T_pK,T_pL}$ and $\mathcal{A}_p$ is the affine tangent translation plane of $\mathcal{P}$ at the point p. We want to determine upper bounds for the possible dimensions of Φ. The next lemma will be the key result for this task.

LEMMA 2.6. *If* $\dim \Psi \geqslant 14$*, then the derivation map* $\mathrm{D}_p \colon \Phi \to \Omega$ *is a closed mapping.*

We prove Lemma (2.6) in several steps. We start with a lemma on subgroups of Lie groups. An *analytic* subgroup H of a Lie group G is a subgroup of G which admits a Lie group structure such that the inclusion map $\iota \colon H \hookrightarrow G$ is a Lie group homomorphism, cp. [18], Section 2, Chap. 9; note that in [18] analytic subgroups are called *virtual Lie subgroups*. A virtual Lie subgroup of a Lie group may not be a closed subgroup. In contrast, a Lie subgroup H of G is a Lie group with respect to the induced smooth structure of G. A subgroup of G is a Lie subgroup if and only if it is closed in G. We introduce a few abbreviations. The torus rank of a Lie group G, i.e. the dimension of a maximal torus subgroup, is written by $\mathrm{rk}\, G$. The centralizer of H in G is denoted by $\mathrm{C}_G H$.

LEMMA 2.7. *Let* H *be an analytic subgroup of a Lie group* G*, and let* T *be a torus subgroup of* H *of rank* r*. If* $r \geqslant \mathrm{rk}\, G - 1$*, then the centralizer* $C_H T$ *is closed in* G*. In particular,* $C_H T$ *is a Lie subgroup of* G.

Proof. A subgroup U of G is closed if and only if every closed one-parameter subgroup of U is closed in G, see Hochschild [14], XVI, Th. 2.4. Assume that there is a closed one-parameter subgroup P of $\mathrm{C}_H T$ which is not closed in G. Then $P \cap T = \mathbb{1}$, because P is not compact, otherwise it would be closed in G.

Moreover, the closure $\overline{P}$ of P in G is compact and connected, [14], XVI, 2.3. Since P is commutative, the closure $\overline{P}$ is commutative as well. The dimension of $\overline{P}$ is greater than dim $P = 1$, because a subgroup of maximal dimension of a connected Lie group coincides with the whole group. The product $B = P \cdot T$ is a direct product, i.e. we have $B = P \times T$, since P commutes with T by hypothesis and P intersects T trivially. Since P is closed in $\mathrm{C}_H T$, we get $\overline{P} \cap \mathrm{C}_H T = P$, whence $\overline{P} \cap T = P \cap T = \mathbb{1}$. Thus we conclude that $\overline{B} = \overline{P \times T} = \overline{P} \times T$ is a compact Abelian Lie subgroup with $\dim \overline{B} > \dim B$. In particular, we have $\overline{B} \cong \mathbb{T}^{r+k}$, where $k \geqslant 2$. This, however, contradicts the fact that the torus rank of G is at most $r + 1$. This proves the Lemma.

LEMMA 2.8. *Let* Ψ *be a Levi subgroup of* Φ, *and let* $\sqrt{\Phi}$ *denote the solvable radical of* Φ. *Then* $\sqrt{\mathrm{D}_p\Phi} = \mathrm{D}_p\sqrt{\Phi}$, $\mathrm{D}_p\Phi = \mathrm{D}_p\Psi \cdot \mathrm{D}_p\sqrt{\Phi}$ *is a Levi decomposition of* $\mathrm{D}_p\Phi$, *and the inequality*

$$\dim \sqrt{\mathrm{D}_p\Phi} \leqslant 6 - \mathrm{rk}\,\Psi$$

holds.

Proof. By the Levi decomposition we write $\Phi = \Psi \cdot \sqrt{\Phi}$ with $\dim(\Psi \cap \sqrt{\Phi}) = 0$, see [19], Chap. 1, 4.1. By Nagami [17], (2.1), this yields $\dim \Phi = \dim \Psi + \dim \sqrt{\Phi}$. The derivation map D_p is a Lie isomorphism between the Lie groups Ψ and $\mathrm{D}_p\Psi$, see [4], Proposition (3.1). Clearly, the image $\mathrm{D}_p\sqrt{\Phi}$ is a solvable normal subgroup with $\dim(\mathrm{D}_p\Psi \cap \mathrm{D}_p\sqrt{\Phi}) = 0$. Since $\mathrm{D}_p\Psi$ is semisimple and Φ is connected, this shows that $\mathrm{D}_p\Phi = \mathrm{D}_p\Psi \cdot \mathrm{D}_p\sqrt{\Phi}$ is a Levi decomposition of $\mathrm{D}_p\Phi$. This proves the first equation. By [3], Theorem (1.6) and [2] Corollary (3.9), the maximal dimension of a closed solvable subgroup of $\mathrm{Aut}(\mathcal{A}_p)_{\mathrm{T}_pK,\mathrm{T}_pL}$ is 6. A maximal solvable subgroup of $\mathrm{D}_p\Phi$ is conjugate to some subgroup of $\mathsf{T} \cdot \sqrt{\mathrm{D}_p\Phi}$, where T is a maximal torus subgroup of Ψ. Using Nagami [17], (2.1) once again, this proves the inequality.

We will need some information about subgroups of the orthogonal group $\mathrm{SO}_8\mathbb{R}$, cp. Hähl [11], 2.8, and [29], 95.12.

LEMMA. 2.9. *Let* K *be a closed connected subgroup of* $\mathrm{SO}_8\mathbb{R}$ *which does not contain a subgroup isomorphic to* $\mathrm{SO}_5\mathbb{R}$. *Then either* K *is isomorphic to one of the groups* $\mathrm{Spin}_7\mathbb{R}$, $\mathrm{U}_4\mathbb{C}$, $\mathrm{SU}_4\mathbb{C}$, *or* $\mathrm{G}_{2(-14)}$, *or* $\dim \mathsf{K} \leqslant 13$ *holds. Moreover, we have* $\mathsf{K} \not\cong \mathrm{U}_4\mathbb{C}$, *if* $\mathrm{rk}\,\mathsf{K} \leqslant 3$.

Proof. By hypothesis, the group $\mathrm{SO}_8\mathbb{R}$ is not contained in K. In particular, we have $\dim \mathsf{K} < \dim \mathrm{SO}_8\mathbb{R}$. Since $\dim \mathrm{SO}_8\mathbb{R} = 28$, this implies that $\dim \mathrm{SO}_8\mathbb{R}/\mathsf{K} \geqslant 7$, or, equivalently, that $\dim \mathsf{K} \leqslant 21$ holds, see Mongomery, Zippin [16], Chap. VI, Th. 2. Thus we have to check which compact groups of dimension at most 21 are subgroups of $\mathrm{SO}_8\mathbb{R}$. According to the Levi decomposition we may write $\mathsf{K} = \Psi \cdot \mathsf{T}$, where Ψ is a semi-simple compact group and $\mathsf{T} \cong \mathbb{T}^r$ is a central torus subgroup. We assume first that K is quasi-simple. Using the classification of quasi-simple Lie groups, see for example [29], Section 95, J. Tits, [30], and recalling that K does not contain a subgroup isomorphic to $SO_5\mathbb{R}$, we infer that K is isomorphic to one of the groups

$$\mathrm{Spin}_7\mathbb{R},\ SU_4\mathbb{C},\ G_{2(-14)},\ \mathrm{Spin}_5\mathbb{R},\ PSU_3\mathbb{C},\ SU_3\mathbb{C},\ SU_2\mathbb{C},\ SO_3\mathbb{R}.$$

Note that a group which is locally isomorphic to $SU_3\mathbb{H}$ has dimension 21, but such a group does not have a faithful representation of dimension less than 12 and hence is not a subgroup of $SO_8\mathbb{R}$. The same argument excludes the group $PSU_4\mathbb{C}$ from being a subgroup of $SO_8\mathbb{R}$. Since the groups that appear on the right hand side of $G_{2(-14)}$ have dimension at most 10, the lemma is proved in the case of a quasi-simple group K. The following table shows the dimensions of real irreducible representations of dimension at most 8, together with their centralizers, see again [29], Section 95, J. Tits, [30].

group	$\mathrm{Spin}_7\mathbb{R}$	$SU_4\mathbb{C}$	$G_{2(-14)}$	$\mathrm{Spin}_5\mathbb{R}$	$PSU_3\mathbb{C}$	$SU_3\mathbb{C}$	$SU_2\mathbb{C}$	$SO_3\mathbb{R}$
dimension	8	8	7	8	8	6	4,8	3,5,7
centralizer	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{R}$

Since a semi-simple Lie group is completely reducible, we get the following centralizers in $GL_8\mathbb{R}$

$\mathrm{Spin}_7\mathbb{R}$	$SU_4\mathbb{C}$	$G_{2(-14)}$	$\mathrm{Spin}_5\mathbb{R}$	$PSU_3\mathbb{C}$	$SU_3\mathbb{C}$	$SU_2\mathbb{C}$	$SO_3\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}\times\mathbb{R}^\times$	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{C}\times GL_2\mathbb{R}$	$\mathbb{H}\times GL_4\mathbb{R}$ or $\mathbb{H}$	$\mathbb{R}\times GL_m\mathbb{R}$

where the possible values of m are $1, 3, 5$.

This shows that a closed subgroup of $SO_8\mathbb{R}$ of dimension at least 14 which does not contain a group of type $SO_5\mathbb{R}$ is isomorphic to one of the groups $\mathrm{Spin}_7\mathbb{R}$, $U_4\mathbb{C}$, $SU_4\mathbb{C}$, $G_{2(-14)}$. Noting that the torus rank of $U_4\mathbb{C}$ is 4, this proves the lemma.

Proof of Lemma 2.6. We use the Levi decomposition $\Phi = \Psi \cdot \sqrt{\Phi}$ of Φ. By [4], Proposition (3.1), the Levi subgroup Ψ is compact and the restriction of the derivation map D_p to Ψ is injective. By Lemma (2.8), $D_p\Phi = D_p\Psi \cdot D_p\sqrt{\Phi}$ is a Levi decomposition of $D_p\Phi$, where $D_p\Psi$ is compact and isomorphic to Ψ. Thus, in order to prove Lemma (2.6), it is sufficient to verify that the radical $\Pi := \sqrt{D_p\Phi} = D_p\sqrt{\Phi}$ is closed in Ω. If $\dim \Psi \geqslant 14$, every closed solvable subgroup of Ψ of maximal dimension (which, of course, is a maximal torus subgroup) is at least 2-dimensional according to the classification of quasi-simple Lie groups, see the tables of J. Tits [30]. Hence, we conclude by Lemma (2.8) that $\dim \Pi \leqslant 6 - 2 = 4$. A quasi-simple compact Lie group Ψ of dimension at least 14 cannot act on a manifold of dimension n less than 5, otherwise $\dim \Psi \leqslant \frac{1}{2}n(n+1) \leqslant 10$

holds by Montgomery–Zippin [16], Chap. VI, Th. 2. Thus, the Levi subgroup $\mathrm{D}_p\Psi$ commutes with the radical Π.

According to [4], Theorem (3.8), the group Ψ is locally isomorphic to some subgroup of $\mathrm{Spin}_8\mathbb{R}$. Consider the two-sheeted covering map $\pi: \mathrm{Spin}_8\mathbb{R} \to \mathrm{SO}_8\mathbb{R}$. Any subgroup Θ of $\mathrm{Spin}_8\mathbb{R}$ is mapped via π onto a subgroup $\pi(\Theta)$, and Θ is an at most two-sheeted covering group of $\pi(\Theta)$. Hence, the torus ranks of $\pi(\Theta)$ and Θ coincide. Recalling that $\mathrm{SO}_5\mathbb{R}$ cannot act on a compact connected projective plane by M. Lüneburg [15], II, Korollar 1 or [29], 55.40, we may apply Lemma (4.9) of [4] in order to obtain that $\pi(\mathrm{D}_p\Psi)$ is isomorphic to one of the groups

$$\mathrm{Spin}_8\mathbb{R}, \quad \mathrm{Spin}_7\mathbb{R}, \quad \mathrm{SU}_4\mathbb{C}, \quad \mathrm{G}_{2(-14)}. \tag{*}$$

The first three groups $\mathrm{Spin}_8\mathbb{R}$, $\mathrm{Spin}_7\mathbb{R}$ and $\mathrm{SU}_4\mathbb{C}$ have torus rank at least 3. By M. Lüneburg [15], II, Satz 2 (see also [29], 55.37), the torus rank of Ω is at most 4. Thus the assumptions of Lemma (2.7) are satisfied for $G = \Omega$, $H = \mathrm{D}_p\Phi$ and some maximal torus subgroup T of $\mathrm{D}_p\Psi$. Hence the centralizer $\mathrm{C}_{\mathrm{D}_p\Phi}\mathsf{T}$ is closed in Ω. By what we have shown above, the radical Π of $\mathrm{D}_p\Phi$ is a subgroup of $\mathrm{C}_{\mathrm{D}_p\Phi}\mathsf{T}$. Since $\mathrm{D}_p\Psi$ is semi-simple, the torus group T is a solvable subgroup of $\mathrm{D}_p\Psi$ of maximal dimension, and thus Π is also the radical of $\mathrm{C}_{\mathrm{D}_p\Phi}\mathsf{T}$. In particular, the group Π is closed in $\mathrm{C}_{\mathrm{D}_p\Phi}\mathsf{T}$. Since $\mathrm{C}_{\mathrm{D}_p\Phi}\mathsf{T}$ is closed in Ω, this shows that Π is closed in Ω, too. This proves Lemma (2.6) in the case, where Ψ is isomorphic to one of the groups $\mathrm{Spin}_8\mathbb{R}$, $\mathrm{Spin}_7\mathbb{R}$ or $\mathrm{SU}_4\mathbb{C}$.

Now let us consider the remaining case $\Psi \cong \mathrm{G}_{2(-14)}$. According to Theorem (2.5), a maximal compact subgroup M of Ω is isomorphic to either $\mathrm{Spin}_8\mathbb{R}$ or $\mathrm{G}_{2(-14)}$. We will show that the centralizer of $\mathrm{D}_p\Psi$ in M is trivial. This is obvious if $\mathsf{M} \cong \mathrm{G}_{2(-14)}$. Hence, we may assume that $\mathsf{M} \cong \mathrm{Spin}_8\mathbb{R}$, and consequently, it is sufficient to verify that the centralizer of $\mathrm{D}_p\Psi$ in $\mathrm{GL}_8\mathbb{R}$ has no nontrivial compact subgroup, see [2], (3.13). Using the second table of Lemma (2.9), we get $\mathrm{C}_{\mathrm{GL}_8\mathbb{R}}(\mathrm{D}_p\Psi) \cong \mathbb{R}^2$. This implies that $\mathrm{C}_{\mathrm{D}_p\Phi}(\mathrm{D}_p\Psi)$ is closed in Ω. As before $\Pi = \sqrt{\mathrm{D}_p\Phi}$ is a (closed) subgroup of $\mathrm{C}_{\mathrm{D}_p\Phi}(\mathrm{D}_p\Psi)$, whence the group Π is closed in Ω. This finishes the proof of Lemma (2.6).

STABILIZER THEOREM 2.10. *Let $\mathcal{P}$ be a smooth 16-dimensional projective plane. Let Φ be a connected closed subgroup of the collineation group of $\mathcal{P}$ which fixes two distinct lines K and L. Let Ψ be a Levi subgroup of Φ. Then exactly one of the following statements is true:*

(i) *$\mathcal{P}$ is isomorphic (as a smooth projective plane) to the octonion plane $\mathcal{P}_2\mathbb{O}$,*
(ii) $\Psi \cong \mathrm{Spin}_8\mathbb{R}$ *and* $\dim \Phi \leqslant 38$,
(iii) $\dim \Psi \leqslant 14$ *and* $\dim \Phi \leqslant 31$.

Proof. Let us assume that $\mathcal{P}$ is not isomorphic to the octonion plane. Then $\mathcal{P}$ is neither a translation plane nor a dual one according to Otte's Theorem (2.4). Thus we have $\dim \Phi_{[p,p]} < 16$, where $p = K \wedge L$. Consequently, we may assume that, say, $\dim \Phi_{[p,K]} < 8$ holds. We have mentioned already that the group Ψ is compact and locally isomorphic to some subgroup of $\mathrm{Spin}_8\mathbb{R}$. We continue our proof with a case by case study depending on the size of $\dim \Psi$.

(1) $\dim \Psi = \dim \mathrm{Spin}_8\mathbb{R} = 28$. Then $\Psi \cong \mathrm{Spin}_8\mathbb{R}$, because the group (P) $\mathrm{SO}_8\mathbb{R}$ cannot act on the tangent plane $\mathcal{A}_p$, see M. Lüneburg [15], II, Korollar 1 or [29], 55.40. Any nontrivial irreducible real representation of $\mathrm{Spin}_8\mathbb{R}$ has dimension at least 8. Thus, the group Ψ acts trivially on the elation group $\Phi_{[p,K]}$, since we have $\dim \Phi_{[p,K]} < 8$. The center of $\mathrm{Spin}_8\mathbb{R}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. These involutions cannot act as Baer involutions on $\mathcal{A}_p$, since $\mathrm{Spin}_8\mathbb{R}$ neither acts trivially on some Baer subplane nor acts nontrivially (as the group $\mathrm{SO}_8\mathbb{R}$) on a Baer subplane. Hence, the center of Ψ contains a central involution ω with center p. By Corollary (4.10) of [2], this involution is not an elation and we may apply Lemma (4.4) of [4] in order to get $\Phi_{[p,K]} = \mathbb{1}$. Thus we obtain

$$\dim \Phi_{[p,p]} \leqslant \dim \Phi_{[p,K]} + 8 = 8$$

by Salzmann [28], (F). According to Theorem (2.5), the group $\mathrm{D}_p\Psi$ is a maximal compact subgroup of Ω, and Lemma (2.2) together with Theorem (2.3) yields

$$\begin{aligned}\dim \Phi &= \dim \mathrm{D}_p\Phi + \dim \ker \mathrm{D}_p \\ &= \dim \mathrm{D}_p\Phi + \dim \Phi_{[p,p]} \leqslant (\dim \Psi + 2) + 8 = 38.\end{aligned}$$

(2) $14 \leqslant \dim \Psi < \dim \mathrm{Spin}_8\mathbb{R} = 28$. By Lemma (2.6), the map $\mathrm{D}_p\colon \Phi \to \Omega$ is closed. Let K be a maximal compact subgroup of Φ that contains the (compact) Levi subgroup Ψ. By [4], (3.8), the group K is locally isomorphic to some closed subgroup of $\mathrm{Spin}_8\mathbb{R}$. Thus we may apply Lemma (2.9) which shows that K is isomorphic to one of the groups $\mathrm{Spin}_7\mathbb{R}$, $\mathrm{SU}_4\mathbb{C} \times \mathbb{T}/\langle -\mathbb{1}\rangle$, $\mathrm{U}_4\mathbb{C}$, $\mathrm{SU}_4\mathbb{C}$, or $\mathrm{G}_{2(-14)}$. Let us first consider the last group. Then $\dim \mathsf{K} = 14$, and we get

$$\dim \Phi \leqslant \dim \mathsf{K} + 2 + \dim \Phi_{[p,p]} \leqslant 14 + 2 + 15 = 31.$$

The groups $\mathrm{Spin}_7\mathbb{R}$ and $\mathrm{SU}_4\mathbb{C}$ both have unique nontrivial 8-dimensional irreducible real representations which map their central involutions α onto $-\mathbb{1}$. Since K is compact and fixes the lines K and L through p, the derivation map D_p maps K bijectively into $\mathrm{GL}_8\mathbb{R} \times \mathrm{GL}_8\mathbb{R}$ if we identify the point space of $\mathcal{A}_p$ with the product $\mathrm{T}_pK \times \mathrm{T}_pL$. If $\mathrm{D}_p\mathsf{K}$ acts trivially on one of the subspaces T_pK or T_pL, then $\mathrm{D}_p\mathsf{K}$ is a homology group with axis, say, T_pK (note: since $\mathrm{D}_p\mathsf{K}$ is compact, it cannot contain an elation with axis T_pK). According to T. Buchanan and H. Hähl [5], a homology group of a locally compact connected translation plane is a closed subgroup of the multiplicative group of the quaternions. Thus, the derivative D_p maps the central involution α onto $-\mathbb{1}$ of $\mathrm{GL}_{16}\mathbb{R}$. In particular, this proves that $\mathrm{D}_p\alpha$ is a homology with center o and so α is a homology with center p. Since we have $\dim \Phi_{[p,K]} < 8$, the group K acts trivially on the elation group $\Phi_{[p,K]}$. Now Lemma (4.4) of [4] applies and we get $\Phi_{[p,K]} = \mathbb{1}$. This provides the inequality

$$\dim \Phi \leqslant \dim \mathsf{K} + 2 + \dim \Phi_{[p,p]} \leqslant 21 + 2 + 8 = 31.$$

(3) $11 \leqslant \dim \Psi \leqslant 13$. Then Ψ is not quasi-simple and we conclude by using the classification of quasi-simple Lie groups that the torus rank of Ψ is at least 3. This yields $\dim \sqrt{\mathrm{D}_p\Phi} \leqslant 6 - 3 = 3$, and as before we end up with $\dim \Phi \leqslant (13+3)+15 = 31$.

(4) $\dim \Psi \leqslant 10$. Then $\dim \mathrm{D}_p\Phi = \dim \Psi + \dim \sqrt{\mathrm{D}_p\Phi} \leqslant 10 + 6 = 16$, and hence we get $\dim \Phi = \dim \mathrm{D}_p\Phi + \dim \Phi_{[p,p]} \leqslant 16 + 15 = 31$. This proves the Theorem.

3. Proof of the Main Theorem

We continue with a reduction result that shows that we may restrict our attention to the case, where the set F_Δ of fixed elements of Δ is given by $F_\Delta = \{p, L\}$ for some flag (p, L), and a Levi subgroup Ψ of Δ is isomorphic to the simple compact group $\mathrm{G}_{2(-14)}$.

PROPOSITION 3.1. *Let* $\dim \Delta \geqslant 39$. *If* $F_\Delta \neq \{p, L\}$, *where* (p, L) *is a flag, or if a Levi subgroup* Ψ *of* Δ *is not isomorphic to the compact exceptional Lie group* $\mathrm{G}_{2(-14)}$, *then* $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$.

Proof. If Δ has more than four fixed elements, then up to duality we may assume that Δ fixes three points o, u, v. Then Δ fixes the two lines $o \vee u$ and $o \vee v$, and we get $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$ by Theorem (2.10). The same is true if Δ fixes two points or two lines. Hence, we may assume that Δ has at most two fixed elements. If Δ fixes an anti-flag, we have $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$ according to Salzmann [25], (2.2). If Δ fixes exactly one element, then $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$ due to Salzmann [27], (C). If Δ contains a normal torus subgroup T, then T is contained in the center of Δ and we can apply [26], (2.1) which shows that if Δ has no fixed elements at all, then Δ is a semi-simple group. Since $\dim \Delta \geqslant 39$, we have once more $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$ by [25], Theorem. Thus we have proved the first part of the proposition.

Now let $\Psi \ncong \mathrm{G}_{2(-14)}$. According to the first part we may assume that $F_\Delta = \{p, L\}$ for some flag (p, L). Then $\mathcal{P}$ or its dual plane is a translation plane by M. Lüneburg, [15], V, Satz, and Otte's Theorem (2.4) on smooth translation planes gives $\mathcal{P} \cong \mathcal{P}_2\mathbb{O}$. This proves the proposition.

According to the last proposition we may assume that $\mathcal{P}$ is a 16-dimensional smooth projective plane, that the group Δ is a closed connected subgroup of $\mathrm{Aut}(\mathcal{P})$ with $\dim \Delta \geqslant 39$ which fixes exactly one flag (p, L), and that a Levi subgroup Ψ of Δ is isomorphic to the compact exceptional Lie group $\mathrm{G}_{2(-14)}$. We proceed by verifying a series of small lemmas. We will always omit the proofs of the dual statements.

(A) *We have* $\dim \Delta = 39$, $\dim \Delta_K = 31$ *for every line* $K \in \mathcal{L}_p \setminus \{L\}$, *and* Ψ *is a maximal compact subgroup of* Δ. *Moreover, the group* Δ *acts transitively on both* $\mathcal{L}_p \setminus \{L\}$ *and* $L \setminus \{p\}$.

Since $\Psi \cong G_{2(-14)}$, we obtain from Theorem (2.10) that $\dim \Delta_K \leqslant 31$ holds. This gives

$$39 \leqslant \dim \Delta = \dim \Delta_K + \dim K^\Delta \leqslant 31 + 8 = 39,$$

whence we have $\dim \Delta = 39$ and $\dim \Delta_K = 31$. The second statement follows immediately from the proof of Theorem (2.10). Let us turn to the last assertion. Since $\mathcal{L}_p \setminus \{L\} \cong \mathbb{R}^8$ is connected and because every orbit of Δ on $\mathcal{L}_p \setminus \{L\}$ is 8-dimensional and hence is open in $\mathcal{L}_p \setminus \{L\}$, we conclude that Δ acts transitively on $\mathcal{L}_p \setminus \{L\}$.

Using the tables of J. Tits [30], we get the following list.

(B) *A compact semi-simple Lie group* Υ *of torus rank at most two is locally isomorphic to one of the groups* $SO_k\mathbb{R}$, $SU_3\mathbb{C}$, *or* $G_{2(-14)}$, *where* $3 \leqslant k \leqslant 5$.

(C) *A proper closed semi-simple subgroup* Υ *of* $G_{2(-14)}$ *has dimension at most* 8.

If Υ is a proper closed subgroup of $G_{2(-14)}$, then $\dim \Upsilon \leqslant 9$ holds according to Theorem 2 of Chapt. VI of [16]. This excludes the group $SO_5\mathbb{R}$ from the list in (B) and hence assertion (C) is proved.

(D) *For every line* $K \in \mathcal{L}_p \setminus \{L\}$ *a Levi subgroup of the stabilizer* Δ_K^1 *is isomorphic to* $G_{2(-14)}$. *In particular, any two Levi subgroups of the stabilizers* Δ_K *are conjugate in* Δ.

Fix some stabilizer Δ_K and let Υ be a Levi subgroup of Δ_K^1. Assume that $\Upsilon \ncong G_{2(-14)}$. Using (C), we obtain $\dim \Upsilon \leqslant 8$, and we infer from Lemma (2.2) and Lemma (2.8) that

$$\begin{aligned} \dim \Delta_K &= \dim D_p\Delta_K + \dim (\Delta_K)_{[p,p]} \\ &\leqslant (\dim \Upsilon - \operatorname{rk} \Upsilon + 6) + 15 \leqslant 14 + 15 = 29 \end{aligned}$$

holds which is a contradiction to (A). Hence assertion (D) is proved.

(E) $\dim \Delta_{[p,p]} = 15$, *and dually*, $\dim \Delta_{[L,L]} = 15$.

The derivation map $D_p \colon \Delta_K \to \Omega$ is closed by Lemma (2.6) and (D). Using the Stabilizer Theorem (2.10), we infer from (A) and (D) that

$$\begin{aligned} 31 &= \dim D_p\Delta_K + \dim \ker D_p \\ &\leqslant (\dim \Psi + 2) + \dim \Delta_{[p,p]} = 16 + \dim \Delta_{[p,p]} \leqslant 31 \end{aligned}$$

holds for every line $K \in \mathcal{L}_p \setminus \{L\}$. This shows that $\dim \Delta_{[p,p]} = 15$.

Furthermore, the inequality above is in fact an equality, whence assertion

(F) $\dim D_p\Delta_K = \dim \Psi + 2$

is true for every line $K \in \mathcal{L}_p \setminus \{L\}$.

(G) *We have* $\dim q^{\Delta_K} \geqslant 7$ *and* $\dim K^{\Delta_q} \geqslant 7$ *for every point* $q \in L \setminus \{p\}$ *and every line* $K \in \mathcal{L}_p \setminus \{L\}$.

Fix some point $q \in L \setminus \{p\}$ and some line $K \in \mathcal{L}_p \setminus \{L\}$. Then, from (A), Lemma (2.8) and the Stabilizer Theorem (2.10), we obtain the inequality

$$\begin{aligned} 31 &= \dim \Delta_K = \dim \Delta_{K,q} + \dim q^{\Delta_K} \\ &\leqslant \dim \mathrm{D}_p \Delta_{K,q} + \dim \ker \mathrm{D}_p|_{\Delta_{K,q}} + \dim q^{\Delta_K} \\ &\leqslant \dim \mathrm{D}_p \Delta_K + \dim(\Delta_{[p,p]})_q + \dim q^{\Delta_K} \\ &\leqslant \dim \Psi + 2 + \dim(\Delta_{[p,p]})_q + \dim q^{\Delta_K} \\ &\leqslant 14 + 2 + 8 + \dim q^{\Delta_K} \\ &= 24 + \dim q^{\Delta_K}, \end{aligned}$$

which gives $\dim q^{\Delta_K} \geqslant 7$.

(H) $\dim \Delta_{[p,L]} = 8$.

Since Δ acts transitively on $L \setminus \{p\}$ by (A), we know that $\dim \Delta_{[q,L]}$ is independent of the point $q \in L \setminus \{p\}$. According to Salzmann [28], Section 0, (G), this implies that $\Delta_{[p,L]}$ is transitive on $K \setminus \{p\}$ for every line $K \in \mathcal{L}_p \setminus \{L\}$. In particular, we have $\dim \Delta_{[p,L]} = \dim K = 8$.

(I) $\dim \Delta_{[q,L]} = 7$ *and, dually,* $\dim \Delta_{[p,K]} = 7$ *holds for every point* $q \in L \setminus \{p\}$ *and every line* $K \in \mathcal{L}_p \setminus \{L\}$.

According to Salzmann [28], (F), we have $\dim \Delta_{[L,L]} \leqslant \dim \Delta_{[q,L]} + 8$ for any point $q \in L$. Thus, we infer from $\dim \Delta_{[L,L]} = 15$ that $\dim \Delta_{[q,L]} \geqslant 7$. If $\dim \Delta_{[q,L]} = 8$ for $q \neq p$, then $\dim \Delta_{[L,L]} = 16$ follows, a contradiction to (E).

PROPOSITION 3.2. *There exists a non-trivial homology* $\omega \in \Delta_{[p]}$.

Proof. We have $\dim \Delta_K = 31$ for any line $K \in \mathcal{L}_p \setminus \{L\}$ by (A). Assertion (D) says that every Levi subgroup Ψ of Δ_K^1 is isomorphic to $\mathrm{G}_{2(-14)}$. The group Ψ is a maximal compact subgroup of Δ (and hence of Δ_K^1 as well), and $\dim \mathrm{D}_p \Delta_K = \dim \Psi + 2$ holds by (F). Hence we may apply Theorem (2.3) in order to obtain that $\mathsf{H} = (\mathrm{D}_p \Delta_K)^1_{[o,L_\infty]}$ is isomorphic to $\mathbb{R}_{>0}$. Since H fixes every tangent space $\mathrm{T}_p G$ for $G \in \mathcal{L}_p$, every collineation of the inverse image $\Theta = \mathrm{D}_p^{-1}\mathsf{H}$ fixes every line through the point p. Conversely, D_p maps every central collineation of $\Delta^1_{[p]}$ into H. This shows that $\Delta^1_{[p]} = \Theta$. By [3], Proposition (1.4), the group $\Delta^1_{[p]}$ can be written as a semi-direct product $\Delta^1_{[p]} = \Delta^1_{[p,A]} \ltimes \Delta^1_{[p,p]}$ for some line $A \notin \mathcal{L}_p$.

In particular, the group $\Delta_{[p,A]}$ is not trivial, because of $\Delta_{[p,p]} = \ker \mathrm{D}_p$ and $\mathsf{H} \neq \{\mathbb{1}\}$. This proves the Proposition.

Fix $K \in \mathcal{L}_p \setminus \{L\}$ and choose a line $M \in \mathcal{L} \setminus \mathcal{L}_p$ such that $M^\Delta \neq \{M\}$. Such a line exists because of $F_\Delta = \{p, L\}$. Due to Proposition (3.2) we can apply a theorem of H. Hähl [9], Corollary 1.3, which yields $M^{\Delta} = M^{\Delta_{[p,p]}}$. Using (E) we get

$$\dim M^{\Delta} = \dim M^{\Delta_{[p,p]}} = \dim \Delta_{[p,p]} = 15$$

and

$$\dim \Delta_M = \dim \Delta - \dim M^{\Delta} = 39 - 15 = 24.$$

The stabilizer Δ_M fixes the antiflag (p, M). Thus we have $(\Delta_M)_{[p,p]} = \{1\!\!1\}$ and from Lemma (2.2) we conclude that

$$\begin{aligned} \dim \mathrm{D}_p \Delta_{K,M} &= \dim \Delta_{K,M} - \dim(\Delta_{K,M})_{[p,p]} \\ &= \dim \Delta_K - \dim M^{\Delta_K} - 0 \geqslant 24 - 8 = 16. \end{aligned}$$

Since $\ker \mathrm{D}_p|\Delta_{K,M} = (\Delta_{K,M})_{[p,p]} = \{1\!\!1\}$, the restriction $\mathrm{D}_p|_{\Delta_{K,M}}$ is an injection.

LEMMA 3.3. *Every Levi subgroup Υ of $\Delta^1_{K,M}$ is isomorphic to* $\mathrm{G}_{2(-14)}$.

Proof. Since $\Psi \cong \mathrm{G}_{2(-14)}$, the group Υ is a subgroup of $\mathrm{G}_{2(-14)}$. If $\Upsilon \ncong \mathrm{G}_{2(-14)}$, then $\dim \Upsilon \leqslant 8$ by (C) and

$$\dim \mathrm{D}_p \Delta_{K,M} \leqslant \dim \Upsilon + 6 - \mathrm{rk}\, \Upsilon \leqslant 8 + 6 = 14$$

follows, which is a contradiction. Thus we have $\Upsilon \cong \mathrm{G}_{2(-14)}$.

According to Lemma (3.3) and the remarks preceding this lemma, we can argue as in Proposition (3.2) in order to get the following corollary. Note that a homology of $\Delta_{K,M}$ with center $q = M \wedge L$ has K as its axis.

COROLLARY 3.4. *There exists a nontrivial homology $\omega^* \in \Delta_{[q,K]}$.*

COROLLARY 3.5. *The stabilizers Δ^1_q and Δ^1_K can be decomposed in the following way: $\Delta^1_q \cong \Delta^1_{q,K} \ltimes \Delta^1_{[q,L]}$ and $\Delta^1_K \cong \Delta^1_{q,K} \ltimes \Delta^1_{[p,K]}$.*

Proof. Proposition (3.2) enables to apply Corollary 1.3 of [9] which yields $K^{\Delta^1_q} = K^{\Delta^1_{[q,q]}}$. In particular, this gives $\dim \Delta_q/\Delta_{q,K} = \dim \Delta_{[q,q]}$. Since $\Delta^1_{[q,q]} = \Delta^1_{[q,L]}$ is a normal subgroup of Δ^1_q that intersects $\Delta_{q,K}$ trivially, the first decomposition is proved. The second decomposition is proved dually.

Now we have collected all the information we need to prove the Main Theorem. Our proof is inspired by the proof of Satz 4 in Chap. VI of [15].

Proof of Main Theorem. By Lemma (3.3), we may assume that the Levi subgroup Ψ of Δ is contained in $\Delta^1_{K,M}$. Let Z be the connected component of the centralizer $C_\Delta \Psi$. Since Ψ fixes both q and K, the subgroups Δ_q^1, Δ_K^1 and $\Delta_{q,K}^1$ are normalized by Ψ. Using [5], VI, Lemma, we thus get

$$\dim \mathsf{Z}_q \equiv \dim \Delta_q (\mathrm{mod}\ 7), \qquad \dim \mathsf{Z}_K \equiv \dim \Delta_K (\mathrm{mod}\ 7),$$
$$\dim \mathsf{Z}_{q,K} \equiv \dim \Delta_{q,K} (\mathrm{mod}\ 7).$$

The fixed set F_Ψ of Ψ is a 2-dimensional subplane of $\mathscr{P}$, the centralizer Z acts almost effectively on F_Ψ, see [15], VII, Satz 5(iv), and p, q, K, L are contained in F_Ψ. Hence we have

$$1 \geqslant \dim q^{\mathsf{Z}} = \dim \mathsf{Z} - \dim \mathsf{Z}_q \quad \text{and} \quad 1 \geqslant \dim K^{\mathsf{Z}} = \dim \mathsf{Z} - \dim \mathsf{Z}_K,$$

because Z fixes p as well as L, and $q \in L \cap F_\Psi$, $K \in \mathcal{L}_p \cap F_\Psi$. If the orbit q^{Z_K} is one-dimensional, we use Corollary (3.5) and (I) to get the contradiction

$$\begin{aligned} 7 \ &= \ \dim \Delta_{[p,K]} = \dim \Delta_K - \dim \Delta_{q,K} \\ &\equiv \ \dim \mathsf{Z}_K - \dim \mathsf{Z}_{q,K} = \dim q^{\mathsf{Z}_K} = 1 (\mathrm{mod}\ 7). \end{aligned}$$

Hence we may assume that the orbit q^{Z_K} (and, dually, the orbit K^{Z_q}) is zero-dimensional. Then Z_K^1 fixes q and, dually, Z_q^1 fixes K. The stabilizer of any quadrangle of a 2-dimensional compact projective plane is trivial, [22], 4.1, whence we have $\dim \mathsf{Z}_q \leqslant 3$ and $\dim \mathsf{Z}_K \leqslant 3$. From [15], VI, Lemma, we have $\dim \mathsf{Z} \equiv \dim \Delta (\mathrm{mod}\ 7)$, which implies that $\dim \mathsf{Z} = 4$. In particular, we have $\mathsf{Z} \neq \mathsf{Z}_q \cup \mathsf{Z}_K$. Choose a collineation $\zeta \in \mathsf{Z} \setminus (\mathsf{Z}_q \cup \mathsf{Z}_K)$. Select some point $u \in K \cap F_\Psi$ distinct from p.

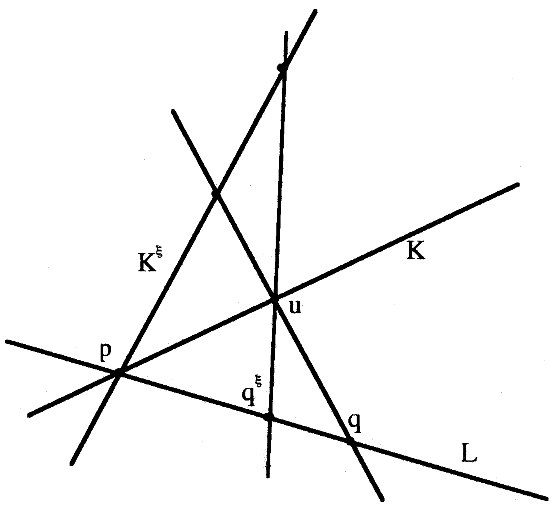

Figure 1.

From $\mathsf{Z}_{K^\zeta}^1 = (\mathsf{Z}_K^1)^\zeta$ we infer that

$$(\mathsf{Z}_{K,K^\zeta})^1 \leqslant \mathsf{Z}_K^1 \cap \mathsf{Z}_{K^\zeta}^1 = \mathsf{Z}_K^1 \cap (\mathsf{Z}_K^1)^\zeta = \mathsf{Z}_{q,K}^1 \cap (\mathsf{Z}_{q,K}^1)^\zeta \leqslant \mathsf{Z}_{q,q^\zeta,K,K^\zeta},$$

whence Z_{u,K,K^ζ} fixes the quadrangle $\{q, q^\zeta, (u \vee q) \wedge K^\zeta, (u \vee q^\zeta) \wedge K^\zeta\}$. However, $\dim \mathsf{Z}_{K,K^\zeta} \geqslant 2$ holds, because of $K, K^\zeta \in \mathcal{L}_p \cap F_\Psi$ and $\dim \mathsf{Z} = 4$. Consequently, we have $\dim \mathsf{Z}_{u,K,K^\zeta} \geqslant 1$, which is a contradiction to Salzmann [22], 4.1. This finishes the proof of the Theorem.

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