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A posteriori estimates for a coupled piezoelectric model

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Abstract: The paper is related to a coupled problem describing piezoelectric effects in an elastic body. For this problem, we deduce majorants of the distance between the exact solution and any approximation in the respective energy class of functions satisfying the boundary conditions. The majorants are fully computable and do not contain mesh dependent constants. They vanish if and only if an approximate solution coincides with the exact one and provide guaranteed upper bounds of errors in terms of the natural energy norm associated with the coupled problem studied.

Keywords: Coupled systems of partial differential equations, piezoelectricity problem, *a posteriori* error estimates.

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Mathematical models arising in the majority of applications are intended to describe so-called multiphysics problems, which involve several equations of different types related to the different processes or phenomena. Piezoelectric models form an important class of such problems with bidirectional coupling between the mechanical and electrical system. These models are highly important for modern technological systems that transform (in a macro or micro scale) mechanical loadings into electric effects and *vice versa*. The first linear mathematical model and the corresponding system of differential equations of an elastic medium taking into account the interaction of electric and mechanical fields was derived by W. Voigt [21]. Later R. Toupin, R. Mindlin, L. Knopoff, S. Kaliski, and J. Petikiewicz presented more advanced models of an elastic medium with polarization (see [2, 8, 9, 19, 20]). The effects evoked by thermal and magnetic fields are considered in [3, 4, 7]. In [1], the authors considered a linear model (without the hysteresis effects) for the interaction of the elastic and electrical fields in a three-dimensional piezoelectric matrix with metallic inclusions.

Typically, numerical solution of multiphysics problems consists of consequent solving the respective equations by a certain splitting scheme that exchanges the data in a suitable order. This is the basic procedure used in lab and industrial environments wherein specialized single-physics solvers have been developed over the years. Difficulties of this approach consists in choosing parameters for the involved solvers that guarantee adequate overall accuracy and efficient use of computational resources. In other words, the challenging problem in quantitative analysis of complicated coupled systems is the reliability of numerical results. To address this contention it is required to understand how to deduce fully computable and guaranteed bounds of the difference between the exact solution and an approximation. This is the main goal of this note, where we derive new *a posteriori* estimates (error majorants) for the coupled system of elliptic partial differential equations motivated by a piezoelectric problem. Our method is based on *a posteriori* error estimates of functional type that was introduced in [11] and developed in [12, 13, 15–17] and many other publications. Error estimates of this type are derived by purely functional methods and provide fully computable measures of the difference between the exact solution to a boundary value problem and an arbitrary function (approxi-

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mation) from the corresponding energy space. They do not attract specific information on the approximate solution (e.g., Galerkin orthogonality, structure of meshes, numerical method used) and do not require extra regularity of the exact solution. As a result, the estimates do not contain mesh dependent constants and are valid for any conforming approximation from the respective class of admissible functions. Moreover, error majorants are nonnegative functionals, which vanish if and only if an approximate solution coincides with the exact one. Therefore, they provide computable measures of the accuracy for a wide spectrum of problems (practical applications of them are systematically discussed in [6]).

The paper is organized as follows. Section 1 introduces the notation and defines the problem. The main result is presented in Theorem 2.1 of Section 2, which states the form of the error majorant. Section 3 presents advanced forms of majorants which are valid for a wider set of functions and contain constants in the so called 'sloshing' inequality.

1 Statement of the problem and notation

Let $\overline{\Omega}$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \in \{2, 3\}$. We denote two primary fields as follows: $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d$ is the vector of elastic displacement, $\varphi : \overline{\Omega} \to \mathbb{R}$ is the scalar electric potential field.

The strain tensor ε is the symmetric part of the displacement gradient:

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right). \tag{1.1}$$

We consider the system of equations describing deformation of a piezocrystallic body in Ω

$$\operatorname{Div} \sigma(\mathbf{u}, \varphi) = f \tag{1.2}$$

$$-\operatorname{div} D(\mathbf{u}, \varphi) = g \tag{1.3}$$

containing the body force vector f and the scalar field of the electric charge density g. Here Div and div denote the divergence operators for the tensor and vector valued functions, respectively, i.e., $\operatorname{Div} \boldsymbol{\tau} = \nabla \cdot \boldsymbol{\tau} = \tau_{ij,j}$ and $\operatorname{div} \mathbf{q} = \nabla \cdot \mathbf{q} = q_{i,i}$. Here and later on the Einstein summation convention of summation over the repeated indices is adopted. The scalar product of vectors is denoted by '$\cdot$' and the scalar product of tensors is denoted by two dots ':'.

The symmetric stress tensor σ and the electric displacement D are coupled via the linear piezoelectric material law:

$$\sigma(\mathbf{u}, \varphi) = \mathbb{L}\varepsilon(\mathbf{u}) + B \cdot \nabla\varphi \tag{1.4}$$

$$-D(\mathbf{u}, \varphi) = K \cdot \nabla\varphi - B^T : \varepsilon(\mathbf{u}). \tag{1.5}$$

In (1.4) and (1.5), $\mathbb{L} = (L_{ijkl})$ is the (forth-order) linear-elastic material tensor, which is subject to the condition

$$c_1^2\,|\varepsilon|^2 \leqslant \mathbb{L}\,\varepsilon : \varepsilon \leqslant c_2^2\,|\varepsilon|^2 \quad \forall \varepsilon \in \mathbb{M}^{d\times d}_{\mathrm{sym}} \tag{1.6}$$

where $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ is the space of symmetric real valued $d\times d$ tensors. We assume that the elements of the elasticity tensor are bounded and possess natural symmetry properties:

$$\mathbb{L}_{ijkm} = \mathbb{L}_{jikm} = \mathbb{L}_{kmij} \in L^\infty(\Omega), \quad i, j, k, m = 1, \dots, d. \tag{1.7}$$

Also, the relations (1.4) and (1.5) contain the (third-order) piezoelectric tensor

$$B = (b_{ijs}), \qquad b_{ijs} \in L^\infty(\Omega)$$

and the (second-order) dielectric material tensor $K = (K_{ij})$, which satisfies the symmetry and ellipticity conditions

$$K_{ij} = K_{ji} \in L^\infty(\Omega) \tag{1.8}$$

$$\gamma_1^2\,|\zeta|^2 \leqslant K\,\zeta \cdot \zeta \leqslant \gamma_2^2\,|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^d. \tag{1.9}$$

The system (1.2)–(1.5) is supplied with the Dirichlét boundary condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_{D,\mathbf{u}} \tag{1.10}$$

for the elastic component of the solution and the condition

$$\varphi = \varphi_0 \quad \text{on } \Gamma_{D,\varphi} \tag{1.11}$$

for the electric component. In general, $\Gamma_{D,\varphi}$ and $\Gamma_{D,\varphi}$ are two different parts of the boundary Γ. On the remaining parts $\Gamma_{N,\mathbf{u}}$ and $\Gamma_{N,\varphi}$, we impose the homogeneous Neumann conditions

$$\sigma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{N,\mathbf{u}} \tag{1.12}$$

$$D \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{N,\varphi}. \tag{1.13}$$

In what follows, we use the spaces $H(\Omega, \mathrm{Div})$ and $H(\Omega, \mathrm{div})$ containing square summable tensor valued and vector valued functions having square summable divergences. Subspaces of these spaces formed by the functions satisfying the homogeneous Neumann conditions are denoted by

$$H^{+}(\Omega, \mathrm{Div}) := \left\{ \boldsymbol{\tau} \in H(\Omega, \mathrm{Div}) \,\Big|\, \int_{\Omega} (\boldsymbol{\tau} : \nabla \mathbf{w} + \mathrm{Div}\boldsymbol{\tau} \cdot \mathbf{w})\, dx = 0 \quad \forall \mathbf{w} \in V_0 \right\}$$

and

$$H^{+}(\Omega, \mathrm{div}) := \left\{ \mathbf{q} \in H(\Omega, \mathrm{div}) \,\Big|\, \int_{\Omega} (q \cdot \nabla \psi + \mathrm{div} q \psi)\, dx = 0 \quad \forall \psi \in M_0 \right\}$$

respectively.

The generalized solution of the problem (1.2)–(1.13) is defined by the system of integral identities

$$c(\mathbf{u}, \mathbf{w}) + b(\varphi, \mathbf{w}) \equiv F(\mathbf{w}) \quad \forall w \in V_0 \tag{1.14}$$

$$-b(\eta, \mathbf{u}) + k(\varphi, \eta) \equiv G(\eta) \quad \forall \eta \in M_0 \tag{1.15}$$

where the solution pair $(\mathbf{u}, \varphi)$ belongs to the sets

$$\mathbf{u} \in V_0 + \mathbf{u_0}, \quad V_0 := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) \big| \, \mathbf{v}|_{\Gamma_{D,\mathbf{u}}} = 0 \right\} \tag{1.16}$$

and

$$\varphi \in M_0 + \varphi_0, \quad M_0 := \left\{ \psi \in H^1(\Omega) \big| \, \psi|_{\Gamma_{D,\varphi}} = 0 \right\}. \tag{1.17}$$

The bilinear forms associated with elasticity, piezoelectric and permittivity tensors and right-hand sides are defined by the relations

$$c(\mathbf{u}, \mathbf{w}) := \int_{\Omega} \varepsilon(\mathbf{u}) : \mathbb{L} : \varepsilon(\mathbf{w})\, d\mathbf{x} = \int_{\Omega} L_{ijkl}\, \varepsilon_{ij}(\mathbf{u}) : \varepsilon_{kl}(\mathbf{w})\, d\mathbf{x} \tag{1.18}$$

$$b(\varphi, \mathbf{w}) := \int_{\Omega} \varepsilon(\mathbf{w}) : B \cdot \nabla \varphi\, d\mathbf{x} = \int_{\Omega} b_{ijs}\, \varepsilon_{ij}(\mathbf{u})\, \varphi_{,s}\, d\mathbf{x} \tag{1.19}$$

$$k(\varphi, \eta) := \int_{\Omega} \nabla \varphi \cdot K \cdot \nabla \eta\, d\mathbf{x} = \int_{\Omega} K_{ij}\, \varphi_{,i}\, \eta_{,j}\, d\mathbf{x} \tag{1.20}$$

$$F(\mathbf{w}) := \int_{\Omega} f \cdot \mathbf{w}\, d\mathbf{x}, \quad G(\eta) := \int_{\Omega} g\, \eta\, d\mathbf{x}. \tag{1.21}$$

since then the bilinear forms are defined first. The solution of the coupled system (1.14)–(1.15) exists and it is a unique element of the set $(V_0 + \mathbf{u}_0) \times (M_0 + \varphi_0)$ (see [18]).

It is not difficult to see that the norm

$$|[\mathbf{u}, \varphi]|^2 := \|\varepsilon(\mathbf{u})\|^2_{\mathbb{L}} + \|\nabla \varphi\|^2_{K} = c(\mathbf{u}, \mathbf{u}) + k(\varphi, \varphi) \tag{1.22}$$

is the natural energy norm associated with our problem. We use it as a suitable measure of the distance to the exact solution.

2 Estimate of the deviation from the exact solution

Assume that $\mathbf{v} \in V_0 + \mathbf{u}_0$ and $\psi \in M_0 + \varphi_0$ are approximations of $\mathbf{u}$ and φ, respectively. Our goal is to deduce a computable upper bound of the norm

$$|[\mathbf{u}-\mathbf{v},\, \varphi-\psi]|^2 := \|\varepsilon(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L}} + \|\nabla(\varphi-\psi)\|^2_K. \tag{2.1}$$

Consider the quantities

$$\mathcal{M}_1(\mathbf{v},\psi,\boldsymbol{\tau}) := \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B\cdot\nabla\psi\|_{\mathbb{L}^{-1}} + \mu_F(\mathbb{L},\Omega,\Gamma_{D,\mathbf{u}})\|f + \operatorname{Div}\boldsymbol{\tau}\| \tag{2.2}$$

and

$$\mathcal{M}_2(\mathbf{v},\psi,q) := \|\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}} + \mu_F(K,\Omega,\Gamma_{D,\varphi})\|g + \operatorname{div}\mathbf{q}\| \tag{2.3}$$

where μ_F are constants in the Friedrichs type inequalities

$$\|\mathbf{w}\| \leqslant \mu_F(\mathbb{L},\Omega,\Gamma_{D,\mathbf{u}})\,\|\varepsilon(\mathbf{w})\|_{\mathbb{L}} \quad \forall \mathbf{w}\in V_0$$

and

$$\|\eta\| \leqslant \mu_F(K,\Omega,\Gamma_{D,\varphi})\,\|\nabla\eta\|_K \quad \forall\varphi\in M_0.$$

Here and later on, $\|\cdot\|$ stands for the L^2 norm of a vector or scalar valued function. The quantities $\mathcal{M}_1$ and $\mathcal{M}_2$ contain only known functions (approximations $\mathbf{v}$ and ψ, and the functions $\boldsymbol{\tau}\in H^+(\Omega,\operatorname{Div})$ and $q\in H^+(\Omega,\operatorname{div})$ that can be viewed as approximations of the exact elastic stress and of the exact flux, respectively).

Theorem below shows that these two quantities majorate the error norm.

Theorem 2.1. (i) *For any* $\mathbf{v}\in V_0+\mathbf{u_0}$ *and* $\psi\in M_0+\varphi_0$ *the combined error norm meets the estimate*

$$|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]|^2 \leqslant \mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\mathbf{q}) \tag{2.4}$$

where $\boldsymbol{\tau}$ *and* q *are arbitrary functions in the spaces* $H^+(\Omega,\operatorname{Div})$ *and* $H^+(\Omega,\operatorname{div})$, *respectively.*
(ii) *The right-hand side of* (2.4) *vanishes if and only if*

$$\mathbf{v}=\mathbf{u},\quad \psi=\varphi,\quad \boldsymbol{\tau}=\mathbb{L}\,\varepsilon(\mathbf{u})+B\cdot\nabla\psi,\quad \mathbf{q}=K\,\nabla\varphi - B^T:\varepsilon(\mathbf{u}).$$

Proof. Let $v\in V_0+u_0$ and $\psi\in M_0+\varphi_0$. We reform (1.14) and (1.15) as follows

$$c(\mathbf{u}-\mathbf{v},\mathbf{w}) + b(\varphi-\psi,\mathbf{w}) = F(\mathbf{w}) - c(\mathbf{v},\mathbf{w}) - b(\psi,\mathbf{w}) \tag{2.5}$$

$$-b\,(\eta,\mathbf{u}-\mathbf{v}) + k\,(\varphi-\psi,\eta) = G(\eta) + b(\eta,\mathbf{v}) - k(\psi,\eta). \tag{2.6}$$

Let $\mathbf{w}=\mathbf{u}-\mathbf{v}$ and $\eta=\varphi-\psi$. By adding equations (2.5) and (2.6), we obtain the norm $|[\mathbf{u}-\mathbf{v},\varphi-\psi]|^2$ on the left-hand side.

On the right-hand side, we have

$$F(\mathbf{w}) - c(\mathbf{v},\mathbf{w}) - b(\psi,\mathbf{w}) + G(\eta) + b(\eta,\mathbf{v}) - k(\psi,\eta) = \int_\Omega (f+\operatorname{Div}\boldsymbol{\tau})\cdot\mathbf{w}\,d\mathbf{x} + \int_\Omega (\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B\cdot\nabla\psi) : \varepsilon(\mathbf{w})d\mathbf{x}$$
$$+ \int_\Omega (g+\operatorname{div}\mathbf{q})\,\eta\,d\mathbf{x} + \int_\Omega (\mathbf{q} - K\nabla\psi + B^T:\ \varepsilon(\mathbf{v}))\cdot\nabla\eta\,d\mathbf{x} \tag{2.7}$$

where $\boldsymbol{\tau}\in H^+(\Omega,\operatorname{Div})$ and $\mathbf{q}\in H^+(\Omega,\operatorname{div})$. The first term is estimated as follows:

$$\int_\Omega (f+\operatorname{Div}\boldsymbol{\tau})\cdot\mathbf{w}\,d\mathbf{x} \leqslant \|f+\operatorname{Div}\boldsymbol{\tau}\|\,\|\mathbf{w}\| \leqslant \mu_F(\mathbb{L},\Omega,\Gamma_{D,\mathbf{u}})\,\|\varepsilon(\mathbf{w})\|_{\mathbb{L}}\,\|f+\operatorname{Div}\boldsymbol{\tau}\|.$$

Analogously,

$$\int_\Omega (g+\operatorname{div}q)\,\eta\,d\mathbf{x} \leqslant \|g+\operatorname{div}\mathbf{q}\|\ \mu_F(K,\Omega,\Gamma_{D,\varphi})\,\|\nabla\eta\|_K.$$

Next, we use the algebraic inequalities

$$\chi : \varepsilon \leqslant (\mathbb{L}\chi : \chi)^{1/2} (\mathbb{L}^{-1}\varepsilon : \varepsilon)^{1/2}$$

and

$$\mathbf{q} \cdot p \leqslant (K\mathbf{q} \cdot \mathbf{q})^{1/2} (K^{-1}p \cdot p)^{1/2}.$$

Hence

$$\int\limits_\Omega (\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B \cdot \nabla\psi) : \varepsilon(\mathbf{w})\,\mathrm{d}\mathbf{x} \leqslant \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B \cdot \nabla\psi\|_{\mathbb{L}^{-1}} \|\varepsilon(\mathbf{w})\|_{\mathbb{L}}$$
$$\int\limits_\Omega (\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})) \cdot \nabla\eta\,\mathrm{d}\mathbf{x} \leqslant \|\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}} \|\nabla\eta\|_K.$$

We estimate (2.7) by the followng two terms:

$$I_1 = \Big(\mu_F(\mathbb{L}, \Omega, \Gamma_{D,\mathbf{u}})\|f + \mathrm{Div}\boldsymbol{\tau}\| + \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B \cdot \nabla\psi\|_{\mathbb{L}^{-1}}\Big) \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbb{L}}$$
$$I_2 = \Big(\mu_F(K, \Omega, \Gamma_{D,\varphi})\|g + \mathrm{div}\mathbf{q}\| + \|\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}}\Big) \|\nabla(\varphi - \psi)\|_K.$$

Then,

$$|[\mathbf{u} - \mathbf{v},\ \varphi - \psi]|^2 \leqslant I_1 + I_2 \leqslant \Big(\mathcal{M}_1^2(\mathbf{v}, \psi, \boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v}, \psi, q)\Big)^{1/2} |[\mathbf{u} - \mathbf{v},\ \varphi - \psi]| \tag{2.8}$$

and we arrive at estimate (2.4).

To prove (ii), we note that vanishing of the right-hand side of (2.4) means that almost everywhere in Ω the following relations hold:

$$\mathrm{Div}\,\boldsymbol{\tau} + f = 0 \tag{2.9}$$
$$\mathrm{div}\,\mathbf{q} + g = 0 \tag{2.10}$$
$$\boldsymbol{\tau} = \mathbb{L}\,\varepsilon(\mathbf{v}) + B \cdot \nabla\psi \tag{2.11}$$
$$\mathbf{q} = K\,\nabla\psi - B^T : \varepsilon(\mathbf{v}). \tag{2.12}$$

From (2.9) and (2.11), it follows that

$$\int\limits_\Omega (\mathbb{L}\,\varepsilon(\mathbf{v}) + B \cdot \nabla\psi) : \varepsilon(\mathbf{w})\mathrm{d}\mathbf{x} = \int\limits_\Omega f \cdot \mathbf{w}\mathrm{d}\mathbf{x} \qquad \forall \mathbf{w} \in V_0. \tag{2.13}$$

Analogously, (2.10) and (2.12) imply

$$\int\limits_\Omega (K\nabla\psi \cdot \nabla\eta - \varepsilon(\mathbf{v}) : B \cdot \nabla\eta)\mathrm{d}\mathbf{x} = \int\limits_\Omega g\eta\mathrm{d}\mathbf{x} \qquad \forall \eta \in M_0. \tag{2.14}$$

Since $\mathbf{v}$ and ψ satisfy the main boundary conditions, (2.13) and (2.14) show that they satisfy system (1.14)–(1.15), which solution is unique. Thus, we conclude that the above functions $\mathbf{v}$ and ψ coincide with $\mathbf{u}$ and φ, respectively. □

Remark 2.1. It is easy to see that

$$\mu_F(\mathbb{L}, \Omega, \Gamma_{D,\mathbf{u}}) \leqslant c_1^{-1}\,\mu_F(\Omega, \Gamma_{D,\mathbf{u}})$$
$$\mu_F(K, \Omega, \Gamma_{D,\varphi}) \leqslant y_1^{-1}\,\mu_F(\Omega, \Gamma_{D,\varphi})$$

where

$$\mu_F^{-1}(\Omega, \Gamma_{D,\mathbf{u}}) = \inf_{\mathbf{w}\in V_0} \frac{\|\varepsilon(\mathbf{w})\|}{\|\mathbf{w}\|} \tag{2.15}$$

and

$$\mu_F^{-1}(\Omega, \Gamma_{D,\varphi}) = \inf_{\eta\in M_0} \frac{\|\nabla\eta\|}{\|\eta\|}. \tag{2.16}$$

Approximate values of $\mu_F(\Omega, \Gamma_{D,\mathbf{u}})$ and $\mu_F(\Omega, \Gamma_{D,\varphi})$ can be found numerically by minimization of the quotients (2.15) and (2.16), respectively. Finding computable and fully guaranteed majorants of these constants is reduced to finding guaranteed lower bounds of the minimal positive eigenvalue of an elliptic operator. This difficult and important problem has been studied by different authors. In the context of Friedrich's type inequalities we refer to [5, 10, 13, 14] and references cited therein.

It is worth adding a concise comment on application of the majorant in combination with finite element approximations. Let $\mathbf{v} = \mathbf{u}_h$ and $\psi = \varphi_h$ be FEM solutions computed on a mesh $\mathcal{T}_h$. The simplest way to apply (2.4) is to reconstruct $\boldsymbol{\tau}$ and $\mathbf{q}$ from the numerical stress $\widehat{\boldsymbol{\tau}}_h = \mathbb{L}\,\varepsilon(\mathbf{u}_h)$ and numerical flux $\widehat{\mathbf{q}}_h = K\nabla\psi_h$. In general, these functions may not belong to $H^+(\Omega, \mathrm{Div})$ and $H^+(\Omega, \mathrm{div})$, respectively. Therefore, a certain averaging (smoothing) procedure is required. A wide spectrum of known post–processing procedures can be used for this purpose (see, e.g., [6, 13] and the references therein). Post–processed functions $\widehat{\boldsymbol{\tau}}_h$ and $\widehat{\mathbf{q}}_h$ preserve continuity of the normal components $\widehat{\boldsymbol{\tau}}_h\mathbf{n}$ and $(\widehat{\mathbf{q}}_h \cdot \mathbf{n})$ along the interelement boundaries. The substitution of $\widehat{\boldsymbol{\tau}}_h$ and $\widehat{\mathbf{q}}_h$ will give a simply computable and guaranteed bound of the error. To make a sharper bound it is necessary to minimize the right-hand side of (2.4) over $\boldsymbol{\tau}$ and $\mathbf{q}$ using $\widehat{\boldsymbol{\tau}}_h$ and $\widehat{\mathbf{q}}_h$ as the initial guess. Since $\mathcal{M}_1^2 + \mathcal{M}_2^2$ is a quadratic functional, this can be done by standard procedures well tested for elliptic boundary value problems (see [6]).

3 Advanced forms of the majorant

The conditions imposed on $\mathbf{q}$ and $\boldsymbol{\tau}$ can be weakened. Below we deduce an advanced form of the majorant, in which the Neumann conditions are to be satisfied in a weak (integral mean) sense. Let

$$\widetilde{H}^+(\Omega, \mathrm{Div}) := \left\{\boldsymbol{\tau} \in H(\Omega, \mathrm{Div}) \mid \{\boldsymbol{\tau}\mathbf{n}\}_{\Gamma_{N,\mathbf{u}}} = 0\right\}$$

and

$$\widetilde{H}^+(\Omega, \mathrm{div}) := \left\{\mathbf{q} \in H(\Omega, \mathrm{div}) \mid \{q \cdot \mathbf{n}\}_{\Gamma_{N,\varphi}} = 0\right\}$$

where $\{q \cdot \mathbf{n}\}_{\Gamma_{N,\varphi}} := |\Gamma_{N,\varphi}|^{-1} \int_{\Gamma_{N,\varphi}} q \cdot \mathbf{n}\, dx$ and $|\Gamma_{N,\varphi}|$ denotes the Lebesgue $(d-1)$-measure of $\Gamma_{N,\varphi}$. Analogously, $\{\boldsymbol{\tau}\mathbf{n}\}_{\Gamma_{N,\mathbf{u}}}$ denotes the vector valued function, which components on $\Gamma_{N,\mathbf{u}}$ are constants equal to mean values of the components of $\boldsymbol{\tau}\mathbf{n}$. Henceforth, we assume that $q \cdot \mathbf{n}$ has a square summable trace on $\Gamma_{N,\varphi}$ and $\tau\mathbf{n}$ is a vector valued function square summable on $\Gamma_{N,\mathbf{u}}$

Then, the right–hand side of (2.7) has a somewhat different form

$$\int_\Omega (f + \mathrm{Div}\boldsymbol{\tau}) \cdot \mathbf{w}\, d\mathbf{x} + \int_\Omega (\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B \cdot \nabla\psi) : \varepsilon(\mathbf{w}) d\mathbf{x} - \int_{\Gamma_{N,\mathbf{u}}} \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{w}\, d\Gamma$$
$$+ \int_\Omega (g + \mathrm{div}\mathbf{q})\, \eta\, d\mathbf{x} + \int_\Omega (\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})) \cdot \nabla\eta\, d\mathbf{x} - \int_{\Gamma_{N,\varphi}} \mathbf{q} \cdot \mathbf{n}\eta\, d\Gamma. \tag{3.1}$$

The integrals over Ω are estimates as in the previous section. For the boundary integrals, we apply the estimates

$$\int_{\Gamma_{N,\varphi}} \mathbf{q} \cdot \mathbf{n}\eta\, d\Gamma \leqslant C_{\Gamma_{N,\varphi}} \|\mathbf{q} \cdot \mathbf{n}\|_{\Gamma_{N,\varphi}} \|\nabla\eta\|_\Omega \leqslant C_{\Gamma_{N,\varphi}} \gamma_1^{-1} \|\mathbf{q} \cdot \mathbf{n}\|_{\Gamma_{N,\varphi}} \|\nabla\eta\|_K \tag{3.2}$$

and

$$\int_{\Gamma_{N,\mathbf{u}}} \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{w}\, d\Gamma \leqslant C_{\Gamma_{N,\mathbf{u}}} \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_{N,\mathbf{u}}} \|\nabla\mathbf{w}\|_\Omega \leqslant C_{\Gamma_{N,\mathbf{u}}} C_K(\Omega) \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_{N,\mathbf{u}}} \|\varepsilon(\mathbf{w})\|_\Omega \leqslant C_{\Gamma_{N,\mathbf{u}}} C_K(\Omega) c_1^{-1} \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_{N,\mathbf{u}}} \|\varepsilon(\mathbf{w})\|_{\mathbb{L}} \tag{3.3}$$

where $C_K(\Omega)$ is a constant in the respective Korn's inequality (which establishes equivalence of $\|\nabla\mathbf{w}\|_\Omega$ and $\|\varepsilon(\mathbf{w})\|_\Omega$ for the vector valued functions in V_0) and $C_{\Gamma_{N,\mathbf{u}}}$ and $C_{\Gamma_{N,\varphi}}$ are constants in the so called 'sloshing' inequality

$$\|\eta\|_{\widetilde{\Gamma}} \leqslant C_{\widetilde{\Gamma}} \|\nabla\eta\|_\Omega \tag{3.4}$$

which holds for any $\eta \in H^1(\Omega)$ with zero mean value on a measurable part of the boundary $\widetilde{\Gamma}$ with positive measure.

In this way, we deduce modified functionals

$$\widetilde{\mathcal{M}}_1(\mathbf{v},\psi,\boldsymbol{\tau}) := \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B\cdot\nabla\psi\|_{\mathbb{L}^{-1}} + \mu_F(\Omega,\Gamma_{D,\mathbf{u}})c_1^{-1}\Big(\|f + \mathrm{Div}\boldsymbol{\tau}\| + C_{\Gamma_{N,\mathbf{u}}}C_K(\Omega)\|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_{N,\mathbf{u}}}\Big) \tag{3.5}$$

and

$$\widetilde{\mathcal{M}}_2(\mathbf{v},\psi,q) := \|\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}} + \mu_F(\Omega,\Gamma_{D,\varphi})y_1^{-1}\Big(\|g + \mathrm{div}\mathbf{q}\| + C_{\Gamma_{N,\varphi}}\|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_{N,\varphi}}\Big). \tag{3.6}$$

We obtain a modified version of Theorem 2.1, where the estimate

$$|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]|^2 \leqslant \widetilde{\mathcal{M}}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \widetilde{\mathcal{M}}_2^2(\mathbf{v},\psi,\mathbf{q}) \tag{3.7}$$

holds for $\boldsymbol{\tau}$ and q in the spaces $\widetilde{H}^+(\Omega,\mathrm{Div})$ and $\widetilde{H}^+(\Omega,\mathrm{div})$, respectively.

As an example, we consider the domain $\Omega = (0,d_1)\times(0,d_2)\times(0,d_3)$, $d_i > 0$, $i = 1,2,3$. This example is motivated by the fact that very often piezoelectric devices are parallelepipeds. Assume that

$$\Gamma_{D,\varphi} = \Gamma_{D,\mathbf{u}} = \{x_3 = 0\}\cup\{x_3 = d_3\}$$

and the Neumann parts of Γ consist of the four lateral faces

$$\Gamma_1 = \{x_1 = 0\},\qquad \Gamma_2 = \{x_1 = d_1\},\qquad \Gamma_3 = \{x_2 = 0\},\qquad \Gamma_4 = \{x_2 = d_2\}.$$

For such a domain the Korn's constant can be estimated.

In [10], it was proved that the constant $C_{\widetilde{\Gamma}}(\Pi)$ for $\Pi := (0,h_1)\times(0,h_2)\times(0,h_3)$ and $\widetilde{\Gamma} = x_1 = 0$ is defined by the relation

$$C_{\widetilde{\Gamma}}(\Pi) = \beta(h_1,h_2,h_3) := (\xi\tanh(h_1\xi))^{-1}, \qquad \xi = \frac{\pi}{\max\{h_2,h_3\}}. \tag{3.8}$$

Let the sets $\widetilde{H}^+(\Omega,\mathrm{Div})$ and $\widetilde{H}^+(\Omega,\mathrm{div})$ be defined by the conditions

$$\{\boldsymbol{\tau}\mathbf{n}\}_{\Gamma_i} = 0,\quad \{\mathbf{q}\cdot\mathbf{n}\}_{\Gamma_i} = 0,\quad i = 1,2,3,4. \tag{3.9}$$

Then, (3.2) is replaced by

$$\begin{aligned}\int_{\Gamma_{N,\varphi}} \mathbf{q}\cdot\mathbf{n}\eta\,\mathrm{d}\Gamma &\leqslant \beta\Big(\frac{d_1}{2},d_2,d_3\Big)\Big(\|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_1}^2 + \|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_2}^2\Big)^{1/2}\|\nabla\eta\|_\Omega\\ &\quad + \beta\Big(\frac{d_2}{2},d_1,d_3\Big)\Big(\|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_3}^2 + \|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_4}^2\Big)^{1/2}\|\nabla\eta\|_\Omega \leqslant R_1(\mathbf{q}\cdot\mathbf{n})y_1^{-1}\|\nabla\eta\|_K.\end{aligned} \tag{3.10}$$

Here the quantity

$$R_1(\mathbf{q}\cdot\mathbf{n}) := \beta\Big(\frac{d_1}{2},d_2,d_3\Big)\Big(\|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_1}^2 + \|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_2}^2\Big)^{1/2} + \beta\Big(\frac{d_2}{2},d_1,d_3\Big)\Big(\|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_3}^2 + \|\mathbf{q}\cdot\mathbf{n}\|_{\Gamma_4}^2\Big)^{1/2}$$

can be considered as a penalty for violations of the homogeneous Neumann condition on $\Gamma_{N,\varphi}$. Analogously, (3.3) is replaced by the estimate

$$\begin{aligned}\int_{\Gamma_{N,\mathbf{u}}} \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{w}\,\mathrm{d}\Gamma &\leqslant \beta\Big(\frac{d_1}{2},d_2,d_3\Big)\Big(\|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_1}^2 + \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_2}^2\Big)^{1/2}\|\varepsilon(\mathbf{w})\|_\Omega\\ &\quad + \beta\Big(\frac{d_2}{2},d_1,d_3\Big)\Big(\|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_3}^2 + \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_4}^2\Big)^{1/2}\|\varepsilon(\mathbf{w})\|_\Omega \leqslant R_2(\boldsymbol{\tau}\mathbf{n})c_1^{-1}C_K(\Omega)\|\varepsilon(\mathbf{w})\|_{\mathbb{L}}\end{aligned} \tag{3.11}$$

where

$$R_2(\boldsymbol{\tau}\mathbf{n}) := \beta\Big(\frac{d_1}{2},d_2,d_3\Big)\Big(\|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_1}^2 + \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_2}^2\Big)^{1/2} + \beta\Big(\frac{d_2}{2},d_1,d_3\Big)\Big(\|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_3}^2 + \|\boldsymbol{\tau}\mathbf{n}\|_{\Gamma_4}^2\Big)^{1/2}$$

can be viewed as a penalty for violations of the homogeneous Neumann condition on $\Gamma_{N,\mathbf{u}}$.

By (3.10) and (3.11), we find that

$$\widetilde{\mathcal{M}}_1(\mathbf{v}, \psi, \boldsymbol{\tau}) := \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B \cdot \nabla\psi\|_{\mathbb{L}^{-1}} + \mu_F(\Omega, \Gamma_{D,\mathbf{u}})c_1^{-1}\Big(\|\,f + \operatorname{Div}\boldsymbol{\tau}\| + C_K(\Omega)R_1(\mathbf{q}\cdot\mathbf{n})\Big) \tag{3.12}$$

and

$$\widetilde{\mathcal{M}}_2(\mathbf{v}, \psi, q) := \|\mathbf{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}} + \mu_F(\Omega, \Gamma_{D,\varphi})y_1^{-1}\Big(\|g + \operatorname{div}\mathbf{q}\| + R_2(\boldsymbol{\tau}\mathbf{n})\Big). \tag{3.13}$$

Hence, for piezoelectric bodies of the parallelepiped form we obtain the estimate

$$|[\mathbf{u} - \mathbf{v},\ \varphi - \psi]|^2 \leqslant \widetilde{\mathcal{M}}_1^2(\mathbf{v}, \psi, \boldsymbol{\tau}) + \widetilde{\mathcal{M}}_2^2(\mathbf{v}, \psi, \mathbf{q}) \tag{3.14}$$

where v and ψ are approximations of the displacement and electric fields, respectively, and $\boldsymbol{\tau}$ and $\mathbf{q}$ (approximations of the stress tensor and flux vector) must satisfy (3.9).

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